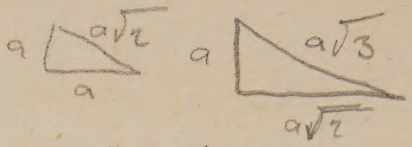
SCHOOL ALGEBRA

BY

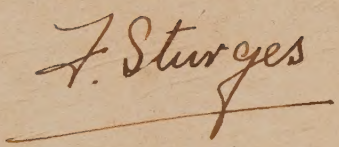
W. E. PATERSON

OXFORD
AT THE CLARENDON PRESS



$\sqrt{2} = 1.414$

$\sqrt{3} = 1.732$



SCHOOL ALGEBRA

BY

W. E. PATERSON, M.A., B.Sc.

MATHEMATICAL MASTER
MERCERS' SCHOOL, LONDON

PART I

OXFORD
AT THE CLARENDON PRESS
LONDON EDINBURGH GLASGOW NEW YORK
TORONTO MELBOURNE CAPE TOWN BOMBAY
HUMPHREY MILFORD
1924

Printed in England

PREFACE

THE main object of this work is to provide a book on modern lines that is suitable for the beginner and can be continued in use for higher forms. In the order adopted and in general treatment the suggestions of the Mathematical Association have been largely followed.

The early chapters are very full in explanation, and are written in simple, even colloquial, language. As far as possible, mathematical terms are not introduced until the processes to which they refer have been rendered familiar by use or are actually going to be used. The usual preliminary list of technical words and their definitions is avoided by placing a general index at the end of the book.

By means of examples, some worked in the text, others set as exercises, the student is led to discover or verify the fundamental laws of Algebra. This experimental method is continued throughout the book. Many examples are set with the intention of leading the student up to a difficulty: thus in some of the early examples such expressions as $3-5$ occur before negative quantities have been mentioned. These examples show the necessity for the next step in the subject, and by giving the student the chance of discovering that step for himself tend to develop the faculty of research.

The necessity of rigorous mathematical treatment is kept in view, and the usual formal proofs are given after the rules have been treated experimentally.

In the middle of Part I there is a Revision Chapter which goes over the early portions again, laying stress on the important points, and introducing new matter which was considered too difficult for the early chapters. There are also several cases of intentional repetition. For example, the importance of the formula $a^2-b^2=(a+b)(a-b)$ is insisted upon two or three times. The author thinks that repetition is more effective than mere reference to a preceding page.

In many places suggestions are made as to how the solutions to questions should be thought out, and how the work should be arranged. On such points of detail each teacher probably has his own views. But it is hoped that the insertion of such matter in a textbook will not only be of help to the inexperienced teacher, but will save the time of the experienced teacher in cases where the pupil has forgotten some previous rule or method of work.

Graphs are treated as illustrations of Algebraic principles or scientific observations, rather than as objects of importance in themselves. But to use them with skill and confidence considerable practice is necessary, and ample provision is made for this.

The practical use of Logarithms is so important that a short treatment of Fractional and Negative Indices is given in Part I, thus enabling this Part to conclude with a chapter on the use of Logarithms.

Horner's Synthetic Method of Division is introduced as an alternative to the usual method. It is surprising that this simple and compact process has not been brought into common use sooner.

The problem is regarded as the important type of example, and most of the various Algebraic operations and processes are introduced as being necessary for the solution of some practical problem. But the more mechanical type of example is not neglected: such examples afford practice in accuracy and neatness, and no student can do good work in Higher Mathematics who has not had experience in dealing with long mechanical Algebraic operations.

Many of the examples are original, but the greater number are taken from recent Examination Papers by permission of the following authorities:—

The Controller of His Majesty's Stationery Office.

The University of Oxford.

The University of Cambridge.

The University of London.

The Joint Matriculation Board of the Scottish Universities.

The Joint Matriculation Board of the Universities of Leeds, Liverpool, Manchester and Sheffield.

The Intermediate Education Board for Ireland.

The Central Welsh Board.

The Delegacy for Oxford Local Examinations.

The Syndicate for Cambridge Local Examinations.

The College of Preceptors.

The author's thanks are due to his colleagues, Mr. F. W. Crampton, M.A., Mr. L. Green, M.A., and Mr. C. B. Wheeler, M.A., for suggestions and help in reading the proofs.

W. E. P.

PREFACE TO SECOND EDITION

A NUMBER of mistakes in the Answers have been corrected in this edition, and my thanks are due to those who have kindly sent me lists of errata. There are also a few verbal alterations in the text.

W. E. P.

October, 1909.

CONTENTS

CHAP.		PAGE
I.	The Use of Letters and Signs	9
II.	The Use of Squared Paper. Power and Index	25
III.	Negative Quantities	32
IV.	Further consideration of the Simple Rules	38
	Miscellaneous Examples	50
V.	Brackets	54
VI.	Easy Functions and their Graphical Representation	63
VII.	Symbolic Representation and Simple Problems	77
VIII.	Equations	83
IX.	Simultaneous Equations. Graphical Solutions	91
X.	Multiplication	102
XI.	Division	111
XII.	Involution and Evolution	120
XIII.	Revision Chapter: — Fundamental Laws. Functional Notation. Formulae. Typical Problems. Identities. Remainder Theorem. Detached Coefficients	130
	Revision Examples	157
XIV.	Factors	162
XV.	Highest Common Factor. Lowest Common Multiple	176
XVI.	Fractions. Equations with Fractions. Revision Examples	180
XVII.	Quadratic Equations. Graphical Solutions. Elementary Theory of Quadratics	201
XVIII.	Graphs of Equations of Second and Higher Degree. Applications	218

CHAP.		PAGE
XIX.	Simultaneous Quadratics. Graphical Solutions	242
XX.	Miscellaneous Problems. Indeterminate Equations	251
XXI.	Indices and Surds	264
XXII.	Common Logarithms and their Practical Use	281
	Miscellaneous Examples	299
	Examination Papers	312
	Index	327

CHAPTER I

THE USE OF LETTERS AND SIGNS

1. Consider the following question :—

If a farmer has 13 *cows in one field,* 16 *cows in a second field, and* 5 *cows in a third, how many cows has he altogether?*

The answer is obtained by adding the three numbers 13, 16, 5, and the work might be arranged thus :—

$$\begin{array}{r} 13 \\ 16 \\ 5 \\ \hline \text{Answer} \quad 34 \text{ cows.} \end{array}$$

If we wish to make quite clear what we have done we use the sign + to show that the number following is to be added to the number before, and the sign = instead of the word 'equals'. The work is then like this :—

$$\begin{aligned} \text{Total number of cows} &= 13+16+5 \\ &= 34 \quad \textit{Answer.} \end{aligned}$$

The sign + is called plus, and is read as plus. 'Six plus five' is written $6+5$; $7+2$ is read 'seven plus two'.

2. The sign — is called **minus**, and means that the number following is to be subtracted from the number before. 'Seven minus three' is written $7-3$, and means that three is to be subtracted from seven.

Examples I a.

The method of working the following examples is to be made clear by using the signs + and −.

1. A baker leaves the shop with 150 loaves. He sells 25 loaves in one street, 48 in another, and 75 in another. How many loaves did he sell and how many did he bring back?

2. There are 35 boys in a form; at the end of the term 15 boys are moved into a higher form and 5 boys leave, while 12 boys are moved up from the form below. How many boys are there in the form in the next term, 3 new boys having come at the beginning of the term?

3. A man has at the beginning of the year 115 books. During the year he buys 27 new books, lends 15, and gives away 17; he himself borrows 4 books. How many books are there in his library at the end of the year?

4. On a journey a man travels 95 miles by train, 12 miles by coach, and walks 5 miles. What was the whole length of the journey?

5. A man's age is 5 years more than his two sons' ages together. He is 55 years old, and one of the sons is 30 years old. How old is the other son?

6. On Sunday at midnight the thermometer stands at 35°; during Monday it rises 15°, it falls 12° on Tuesday, and rises 6° on Wednesday. What is the reading of the thermometer at midnight on Wednesday?

7. A man starts out with £6 in his pocket; he pays bills to the amount of £5, cashes a cheque for £10, and loses a sovereign. How much has he when he returns home?

8. There are four gallons of beer in a cask; if three gallons of water are poured in and two gallons of the mixture drawn out, how much is there then in the cask?

9. In a family of 5 brothers, the youngest is 3 years old, the next 5 years old, the third is as old as the other two together, the fourth is 10 years old, and the eldest is 15 years old. What is the sum of all the ages?

10. After 214 people had left a meeting there were 356 people left. How many were there at first?

11. At the end of a year a merchant finds that his property is worth £12055; during the year he had gained £517. What was the value of his property at the beginning of the year?

3. The question in § 1 would be done the same way whatever the actual number of cows might be in the different fields. In all cases we should find the total number by adding the several numbers. To show this we use letters instead of numbers. Instead of saying '*A farmer*

has a certain number of cows in one field, a certain number in a second field, and another number in a third field', we may say '*A farmer has* x *cows in one field,* y *cows in a second, and* z *cows in a third'*.

If we are then asked how many cows are there altogether, the answer is $x+y+z$. This means that when we know what numbers x, y, z stand for, we are to add those numbers.

In many questions some of the numbers may have to be added and some to be subtracted, and in such cases care must be taken that the proper sign + or − is placed before each quantity. Quantities connected by the signs + or − are called terms.

4. In working examples it is not only necessary to get the right answer but also to show quite clearly how the answer is obtained.

All work must be shown ; no work should be done on any other piece of paper.

Make it quite clear what the answer is.

Example I. *A dealer has* 1000 *sacks of potatoes for sale ; he takes orders for* 215, 332, 120, *and* 426 *sacks. How many more sacks must he buy in order to carry out his orders?*

The dealer takes orders for 215 + 332 + 120 + 426 sacks.

Total number of sacks required = 1093.

He has 1000 sacks.

$$\begin{array}{r} 215 \\ 332 \\ 120 \\ 426 \\ \hline 1093 \end{array}$$

∴ Number of sacks he must buy = 1093 − 1000.
= 93 *Answer.*

Example II. *A dealer has a stock of* x *sacks of potatoes ; he takes orders for* a, b, *and* c *sacks. How many sacks will he have left?*

Number of sacks left = $x-a-b-c$.

Examples I b.

1. A boy has $6d.$ in one pocket and $9d.$ in his money-box; he buys a book for $10d.$ How much has he left?

2. Write out the previous question, using x, y, z instead of 6, 9, 10. What is the answer now?

3. A man walks x miles due east, then y miles west; how far is he now from his starting-point? What distance has he walked altogether?

4. What are the answers to example 3 when $x = 7$, $y = 4$?

5. Make up a question to which the answer shall be $x - y + z$.

6. Find the value of $p - q + r$ when

(i) $p = 3$, $q = 2$, $r = 5$; (iii) $p = 23$, $q = 14$, $r = 0$;
(ii) $p = 13$, $q = 21$, $r = 25$; (iv) $p = 114$, $q = 57$, $r = 178$.

7. When $x = 12$, $y = 3$, $z = 5$, find the value of

(i) $x + y + z$; (iii) $x + y - z$;
(ii) $x - y + z$; (iv) $x - y - z$.

8. There are three forms in a school; in the first form there are x boys, in the second y boys, and there are p boys in the school altogether; how many boys are there in the third form? What does the answer become when $p = 57$, $x = 15$, $y = 21$?

9. Two men form a partnership; one put £x into the business, the other £y. During the first year they spend £x and receive £700. How much have they at the end of the year?

10. An article weighs x lb.; when it is being weighed it is found that a 10 lb. weight will not counterbalance it. What additional weight must be placed with the 10 lb. weight so that there may be an exact balance?

11. If in question 10 it were found that the 10 lb. weight was too much, what weight would have to be added to the other side to get the exact balance?

12. An article, weighing x lb., together with a weight of 4 lb. exactly counterbalances two weights of 10 and 2 lb. What is the value of x?

13. A fish weighed 12 lb.; the head weighed x lb., the body 6 lb.; what did the tail weigh?

14. A field is x yards long and y yards wide; what length of fence is required to surround it?

15. What length of tiling must be used to surround a bed of flowers which is x yards wide and twice as long?

5. Signs of Multiplication.

Example I. *If each man in a cricket eleven makes* 8 *runs and there are* 8 *extras, what is the total for the innings?*

Here there are twelve 8's to be added; in other words the total is twelve times 8, which equals 96.

The process of finding the sum of a number of equal quantities is called **multiplication**, and is usually denoted in Arithmetic by the symbol $\times$, and sometimes by a full stop, e.g. 7×8, 7.8. These both mean 7 multiplied by 8 or 7 times 8. The full stop must not be confused with a decimal point, which is placed higher up. The result of multiplication is called the **product**.

Example II. *If each man in a cricket eleven makes* x *runs and there are* x *extras, what is the total for the innings?*

The total is twelve times x, and is written $12x$.

Thus in Algebra we need use no sign for multiplication, but may simply write the quantities one after the other. If one of the quantities is a number and the other a letter, then the number is called the **coefficient** of the letter; thus in $7x$, 7 is the coefficient of x.

If both the quantities are letters, either of them may be called the **coefficient** of the other.

Notice that the letter x, or a, or y, or any single letter, may be regarded as having coefficient 1, i. e. x is the same as $1x$. But the coefficient 1 is never actually written; it is always left out or understood.

Quantities which are multiplied together are called **factors** of the result. Thus in $9abc$ the factors are 9, a, b, c; but in $a+b+c$, a, b, and c are terms (§ 3).

Multiplication by 0.

Example III. *If no pennies are given to each of* 100 *boys, how much money is given away?*

Method I. Each boy receives 0 pence.

$\therefore$ 100 boys receive 100×0 pence.

Method II. Each boy receives no pennies.

$\therefore$ no money is given away.

i. e. 100 boys receive no pennies.

Comparing the two answers it is seen that $100 \times 0 = 0$.

Similarly it is seen in all cases that when any number is multiplied by nothing the product is nothing. If $x = 0$, then $7x = 0$, $8x = 0$, $ax = 0$, $10000x = 0$, &c.

6. Signs for Division.

Example I. *If a sum of* 16 *shillings is divided among* 4 *people so that they have equal shares, how much will each receive?*

To find the value of one share the whole amount must be divided by 4. The sign $\div$ means 'divided by', so that the work of this example can be put down thus:—

Amount each person will receive $= 16s. \div 4.$

$= 4s.$ *Answer.*

Very often we show that one quantity is to be divided by another by placing the first over the second with a line in between; thus $\frac{16}{4}$ means 16 divided by 4.

Both signs can be used with letters, so that $\frac{a}{3}$, $b \div 3$, $\frac{x}{a}$ mean respectively a divided by 3, b divided by 3, x divided by a. Sometimes instead of $x \div a$, x/a is used.

Example II. *A man left home with* £x *in his pocket; he spent* £y, *and then divided the rest equally among* 15 *boys. How much did each receive?*

After shopping he had $x-y$ pounds left.

Hence he divided $x-y$ pounds among 15 boys.

$\therefore$ each boy received $\frac{x-y}{15}$ pounds.

Example III. *A man had* £x; *he gave* £5 *to each of his servants and then had* £y *left. How many servants had he?*

He gave away $x-y$ pounds.

He gave 5 pounds to each servant.

$\therefore$ he had $\frac{x-y}{5}$ servants.

Examples Ic.

1. What is the value of x horses at £a each?

2. If y lb. of sugar cost £x, how many pence will 1 lb. cost?

3. If 3 cheeses cost $3p$ shillings, what will x cheeses cost?

4. Two places, A and B, are x miles apart; how long will it take to walk from A to B at the rate of 4 miles per hour?

5. In a certain library there are two rooms; one room contains six shelves with fifty books on each shelf, whilst the other contains eight shelves with forty-two books on each shelf. How many books are there in the library?

6. In a certain library there are two rooms; one room contains x shelves with 20 books on each shelf, whilst the other contains eight shelves with y books on each shelf. How many books are there in the library?

7. What does the answer to No. 6 become in each of the following cases?

(i) $x = 5, y = 18$; (ii) $x = 20, y = 0$; (iii) $x = 30, y = 25$.

8. If $a = 1, b = 2, c = 3, d = 0$, find the value of

$$8ab + 3bc + abc - 7bcd.$$

9. A man had x horses, he kept y and sold the remainder. He received z pounds; how much did he get for each horse?

10. What does the answer to No. 9 become when

$$z = 400, x = 25, y = 5?$$

11. If $x = 3, y = 4, z = 0$, find the value of

$$5xy + \frac{8x}{y} - 9xyz.$$

12. If 1 lb. of butter costs x pence, how many shillings will a hundredweight cost?

13. From a stick x inches long, p pieces, each of length y inches, are cut; how long is the piece that is left?

14. In a race A runs x yards a second, and B y yards a second; if B has a yards start, in how many seconds will A overtake B?

15. During the cricket season a boy scores x, y, $2x$, $3y$, $3x$, $2y$ runs in successive innings. What is his average?

16. If $x = 1, y = 2, z = 3$, find the value of $2xy + 3yz - 4xy$.

17. When $x = 1, y = 2$, it is found that $2x + 3y - 4z$ equals 8; what is the value of z?

18. A man with x nephews gave them 5 shillings each and then had 7 shillings left in his pocket. How many shillings had he at first?

19. What is the meaning of $3x$? Find the value of

$$3x+4x-2x.$$

20. A man walked from a place P to a place Q and then to a place R. The distance from P to Q was x miles and it was three times as far from Q to R. He walked altogether 12 miles; how far was it from P to Q?

7. Example. *There are* x *boys in the 1st form, twice as many in the 2nd form, and four times as many in the 3rd form. How many are there in the three forms together?*

Number of boys $= x+2x+4x$
$= 7x.$

For just as 1 penny + 2 pennies + 4 pennies = 7 pennies
so $1x \quad +2x \quad +4x \quad = 7x.$

Terms which contain the same letter are called like terms.

Rule for adding like terms. Add up the coefficients of the several terms. The result is the coefficient in the answer, e. g. $3y+4y+6y = 13y$, since $3+4+6 = 13$.

Rule for adding any number of algebraical expressions. For convenience arrange the expressions one under the other, so that like terms come in the same column (just as £ *s. d.* are arranged in compound addition), then add up the several columns of like terms, and insert + signs between the several results.

The result of adding quantities is called their **sum.**

Example. *Add together*

$$2a+3b+4c,\ 3a+b+2c,\ a+2b+4c.$$

$$\begin{array}{r} 2a+3b+4c \\ 3a+\ b+2c \\ \ a+2b+4c \\ \hline 6a+6b+8c \end{array} \quad \text{Sum.}$$

Remember that the coefficient 1 is understood in such terms as a, $+b$, and must be added in with the other coefficients.

8. Example I. *A tradesman begins the day with £18; he sells goods to the value of £10 and is paid £12 owing to him, he loses £5, pays £8, and lends £4. How much has he at the end of the day?*

Amount at the end of the day $= 18+10+12-5-8-4$ pounds
$= 40-17$ pounds
$= £23$ *Answer.*

In this example the actual order of the various receipts and payments does not affect the result; we take no account of the order but add up all the receipts, add up all the payments and losses, and then subtract.

Example II. *A tradesman begins the day with £4y; he sells goods to the value of £3y and is paid £y owing to him; he loses £2y, pays £y, and lends £3y. How much has he at the end of the day?*

Exactly as in the arithmetical example we add up the quantities increasing his money, add up the quantities decreasing his money, and then subtract.

Amount at end of day $= 4y+3y+y-2y-y-3y$ pounds
$= 8y-6y$ pounds
$= 2y$ pounds.

Hence we deduce an extension of the addition rule.

Rule. If in a number of like terms some are to be added and some subtracted, first add all those that are to be added, then those that are to be subtracted, and subtract the second sum from the first.

The result is called the **algebraic sum** of the quantities.

Examples I d.

Add together:

1. $3x+4y-z$, $2x-y+2z$, $5x-2y+4z$.
2. $+2a-3b-4c$, $3a-2b+5c$, $4a+7b-c$.
3. $3a+2b-c$, $2a-b$, $-2c+3b+2a$, $a+b-c$.
4. $2p+q$, $5p+4q-r$, $4r-2q+3p$, $3q$.
5. $3x-4y+5z$, $-x+5y-2z$, $2x+2y+4z$.

Simplify:

6. $4x-2y+3z+4z-2x+3y-8x+y-4z+9x$.
7. $3a+4b-2a+9b+3a-c+2a-4b+6c+4a$.

8. $2ab+3bc-4ca-3ab+5ca-2bc+4ab.$
9. $3p-4x+2a-4p-3a+5x-2x+2p+3a+6x$
10. $2b+2c-3a+4a+5b-c-a-4b+3c.$

If $a=3$, $b=2$, $c=1$, find the value of

11. $2ab+3bc-4ac.$
12. $3a\times 2b\div c+4b\times 3a\div 2c.$
13. $3a+4b-2a+2c-2b+3a-c.$
14. $c-2a+3b-4c+8b-3a.$
15. $4abc-2ab-3bc-ab.$

If $x=3$, $y=0$, $z=4$, find the value of

16. $3x+\frac{y}{z}-2z.$
17. $xy+4yz+3zx-4x+5y+6z-2xyz.$
18. $3x-4z+2y-x+2z-3y+17x-4z+3.$

9. Use of Brackets.

Example I. *A purse contains 3 compartments; each compartment contains one sovereign, one shilling, and one penny. What is the total value of the money in the purse?*

Method I. Value of money in each compartment = one sovereign + one shilling + one penny.

∴ value of money in purse = 3 times the result obtained by adding one sovereign, one shilling, and one penny.

Which we express thus:

value of money in purse = 3 (1 sovereign + 1 shilling + 1 penny).

This means that the quantities in the brackets ( ) are first to be added and the result is to be multiplied by 3.

Method II. There are altogether 3 sovereigns, 3 shillings, and 3 pennies.

i. e. value of money in purse = 3 sovereigns + 3 shillings + 3 pennies.

Comparing the two methods we see that

3 (1 sovereign + 1 shilling + 1 penny) = 3 sovereigns + 3 shillings + 3 pennies.

Example II. *On each day of the week, except Sunday, a man receives 7 shillings and pays away* 6*s.* 2*d. How much does he save per week?*

Method I. Amount saved per day $= 7s. - 6s.\ 2d.$
$\therefore$ amount saved per week $= 6\ (7s. - 6s.\ 2d.)$.

Method II. Total amount received $= 6 \times 7s.$
Total amount spent $= 6 \times 6s.\ 2d.$
$\therefore$ amount saved per week $= 6 \times 7s. - 6 \times 6s.\ 2d.$

Comparing the two methods we see that

$$6\ (7s. - 6s.\ 2d.) = 6 \times 7s. - 6 \times 6s.\ 2d.$$

Example III. *A traveller is at a height of* 10000 *feet; the next day he descends* 2000 *feet and ascends* 500 *feet. What is the height now?*

Method I. Since he first descends 2000 feet and then ascends 500 feet, he descends (2000 − 500) feet in the course of the day.
But he started the day at a height of 10000 feet.
$\therefore$ at the end of the day the height is 10000 − (2000 − 500) feet.

Method II. When he has descended 2000 feet the height is 10000 − 2000 feet.
He now ascends 500 feet.
$\therefore$ final height = 10000 − 2000 + 500 feet.

Hence we see that $10000 - (2000 - 500) = 10000 - 2000 + 500$.

Example IV. *A traveller is at a height of* 10000 *feet; on each of three successive days he descends* 2000 *feet and ascends* 500 *feet. What is the height at the end of those three days?*

Method I. During each day he descends (2000 − 500) feet.
$\therefore$ in three days he descends 3 (2000 − 500) feet.
$\therefore$ at the end of the three days the height is 10000 − 3 (2000 − 500) feet.

Method II. During the three days he descends 6000 feet, and ascends 1500 feet.
$\therefore$ at the end of the three days the height is 10000 − 6000 + 1500 feet.

Hence we see that $10000 - 3\ (2000 - 500) = 10000 - 6000 + 1500$.

There would be no difference in the working of the previous examples if we had used letters instead of numbers.

Examples I e.

Each of the following examples should be worked by the two methods: (i) using brackets, (ii) not using brackets.

1. A man earns x shillings a week and spends y shillings a week. How much has he saved at the end of a year?

2. A man begins the year with £100. He earns £10 a month and spends £a a month. How much has he at the end of the year?

3. If, in question 2, a is greater than 10, how much has he at the end of the year?

4. In a farmyard there are three sheds; in each shed there are x fowls and y ducks. What is the total number of birds?

5. On each of four days of a week a merchant gained £x and lost £y. On each of the remaining week-days he lost £x and gained £y. If x is greater than y, what were his gains for the whole week?

6. A gardener earns p shillings a week and his son q shillings a week; how many shillings will they together earn in 5 weeks?

7. There are 4 bookshelves in a room; each of them contains 10 English books, 8 German books, and 7 French books. How many books are there in the room?

8. A man has 5 children; each child has x sons and y daughters. How many grandchildren has the man?

10. We see from the preceding examples * that we may omit brackets if we are careful to observe the following rules:

(i) If there is a + sign only before the bracket, the quantities inside the bracket are written down unchanged.

$$\text{e.g. } +(a-2b+3c) = +a-2b+3c.$$

If no sign is written before the first term inside a bracket the sign + is understood.

(ii) If there is a factor preceding or following the bracket, that factor multiplies each of the terms inside the bracket;

* The student must be warned that a rule cannot be proved by considering a particular case; we are, however, led to see what the rules are by considering several special cases; in later chapters proofs of these rules will be given.

if the factor or bracket is preceded by the sign +, the signs of the products are the same as the signs of the corresponding terms inside the bracket.

$+7(x-2y) = +7x-14y$ and $+(x-3y)a = ax-3ay$.

(iii) If there is a − sign only before the bracket, the sign of each quantity inside the bracket must be changed when the brackets are omitted; all the + signs become −, all the − signs +; e. g. $-(2x+3y-4z) = -2x-3y+4z$.

(iv) If a factor precedes the bracket, and a − sign precedes that factor, then each term inside the bracket must be multiplied by the factor, but the sign of each product is opposite to the sign of the corresponding term in the brackets; e. g. $-3(4a-2b+5c) = -12a+6b-15c$.

The process of omitting brackets is usually called 'removing brackets'.

Examples I f.

EXAMPLES FOR PRACTICE IN REMOVING BRACKETS.

Remove the brackets from the following expressions, and simplify the results as far as possible:

1. $2(2x+y-z)$.
2. $3a(2x-y+z)$.
3. $3x-2(x-y)$.
4. $6-3(5-4)$.
5. $7+2a(b-c)$.
6. $7x-3(x-y)+2(y+x)$.
7. $(x+y)+(x-y)$.
8. $(x+y)-(x-y)$.
9. $(3a-b)+5(b-a)$.
10. $3a-4(x-2y)$.

11. In an examination A gets m marks, B gets n marks, and C gets twice the difference between A's and B's marks. How many marks has A more than C?

12. A and B have each 30 shillings and C has 15 shillings. A gives x shillings to B and three times as many to C, while C gives y shillings to B. How many shillings has each then? How many have they altogether?

13. Add together $(3a-2b)x+(a-3b)$ and $(3a+2b)x-(a+3b)$.

14. Add $x+2(y-z)$, $y-2(z-x)$, $z-2(x+y)$.

15. Simplify $3(4+5-1)-2(2+3-4)+4(3+4-5)$.

16. Find the sum of $2a-(5c+3b)$ and $4b-(a-3c)$; and subtract $(a-3c)$ from the result.

17. If $a = 3$, $b = 5$, $c = 0$, and $d = 1$, find the value of

$$\frac{(4a-3c)d}{(3b-5d)} + \frac{5abc}{9(bc+ad)} - \frac{(2a-4d)}{3b+c}.$$

18. Add $2(a-3b-2c)$, $3(b-4c+2a)$, and $4(2c-a+5b)$.

Examples I.

Miscellaneous.

1. An article costs £a; what is its value in shillings? How many could be bought for x shillings?

2. From $7a+2b+3c$ take $3a+b+2c$.

3. Add together $2a+4b-2c$, $3a-5b+6c$, $2b-4a+c$.

4. Simplify $3(2a-b)+2(a+2b)-4(a-2b)$.

5. An army consists of x regiments, each regiment contains y companies of z men each. How many men are there in the army?

6. Three boys spend their money on cakes. The first has a pounds, the second b shillings, and the third c pence. If each cake costs x pence, how many cakes can they buy?

7. What is the difference between a term and a factor? Write down two expressions, one containing three terms the other containing three factors.

8. A boy is x years old in the year 1908; how old will he be in 1917? When will he be 21 years old?

9. What must be added to $2a+b+4c$ to make $8a+4b+5c$? Verify the answer by putting $a = 1$, $b = 2$, $c = 5$.

10. A farmer sold a fowls at b pence each, b lambs at a pounds each, and with the money so obtained bought $(a+b)$ geese at c shillings each. How much money had he left?

11. If $p = 0$, $q = 1$, $r = 3$, find the value of

$$p(q-r)+q(r-2p)+3r(p+4q).$$

12. To each of 10 old women there was given a present consisting of a lb. of tea and b lb. of sugar; what was the total weight given away?

13. Simplify $2(-x+y)-3(x-y)+4(3x+2y)$.

14. There are three consecutive numbers of which the middle one is x; what are the others?

15. A boy returned to school with 15s. in his pocket; an uncle gave him x shillings, his father five shillings, and his mother a florin. What is the value of x?

16. Multiply $3a+4b+5c$ by $2x$.

17. A boy walks a miles b furlongs on each of three days in the week, $2b$ miles a furlongs on each of three other days, and b miles $2a$ furlongs on the remaining day. How many furlongs did he walk that week?

18. Find the sum of $2x+3y-z$, $3x-4y+6z$, $-x-y+5z$, and $-2x+6y-7z$.

19. If $a=2$, $b=3$, $c=0$, find the value of

$$3a+4b-2a(3b-4c)+3c(14a-2b).$$

20. A boy is x years old in 1908; in what year was he born? When will he be $2x$ years old?

21. Think of a number, double it, add 5, subtract 3, divide by 2, take away the number thought of and the remainder is always 1. Prove this to be true by taking x as the number

22. When $a=3$, $b=5$, $c=7$, prove that

$$a(b+2c)+3b(c+2a)+3c(a+2b)=7ab+5ac+9bc.$$

23. What number is it (call it x) which when 5 is added to it is 7 less than double the number?

24. What is the remainder when $4a+5b+8c$ is diminished by $a+b+c$?

25. How many times will a bicycle wheel turn round in going x miles if it turns round once in every y yards?

26. When two numbers are added their sum is 96; one of them is x, what is the other?

27. If x is an even number, what is the next odd number?

28. Find the value of

$$3+5(6-\tfrac{12}{4})-3(2+2.4)+12(0+\tfrac{9}{6}).$$

29. Find the value of $(2a+3b-4c)-(a+2b-5c)$

30. Subtract $a+2b-5c$ from $2a+3b-4c$.

31. Remove the brackets and simplify

$$-5(a-b-c)+30(b-c-a)+6c-4(c+b-a).$$

32. A regiment originally contained x men; after y men had been killed, the regiment was divided into p companies. How many were there in each company?

33. If 19 cricket balls at x shillings each cost £4. 15*s*. what is the value of x?

34. If I have in my purse a pounds b shillings, and I spend c shillings and c pence, how many pence have I left?

35. Write down five consecutive numbers so that x shall be the middle number of the five. What is their sum?

36. Distinguish clearly between the meanings of the algebraical expressions $\frac{a+b}{c}$, $a+\frac{b}{c}$, $\frac{ab}{c}$; and give the value of each expression when $a=1$, $b=2$, $c=3$.

37. If a train travels a yards in b seconds, how long will it take to travel 1 mile?

38. If a train travels at the rate of a miles an hour, how many hours will it take to travel b miles and how many seconds will it take to travel b yards?

39. Add together $a+(2b+3c)$, $2a-(b+2c)$, $b-(c+a)$, and $c-(a+b)$.

40. What must be subtracted from $5y+2x+8$ so as to leave $3y+2x+7$?

Find the value of $\frac{n}{2}(a+l)$ in the following cases:

41. $n=17$, $a=5$, $l=17$.
42. $n=5$, $a=7$, $l=109$.
43. $n=16$, $a=17$, $l=95$.
44. $n=20$, $a=1$, $l=6$.

Find the value of $\frac{Prt}{100}$ when

45. $P=115$, $r=4$, $t=5$.
46. $P=725$, $r=6$, $t=2$.
47. $P=136$, $r=5$, $t=5$.

48. Explain the words term, factor, coefficient, quotient; and give examples.

49 How many shillings change will there be out of a sovereign after paying a bill for a pints of milk at b pence per quart and c dozen eggs at the rate of d shillings a dozen?

50. A man starts with a sovereigns b shillings and c pence in his purse. How much money has he in all, expressed in pounds? Supposing him to pay away d sovereigns and f pence and to receive e sixpences, how much money has he left, expressed in pence? How many coins had he at first?

51. A man's salary is £x a day and he spends £y a day. How much does he save in z days? How long will he take to save £100?

52. Simplify $(2a-\frac{3}{2}b+\frac{1}{3}c)+(-\frac{1}{2}a+2b-\frac{1}{2}c)-(\frac{3}{2}a-b+2c)$.

53. From $2b-(c-a)$ take $2c-(a-b)$.

54. x and y are two numbers; express algebraically the two facts that their sum is 620 and their product 3684.

55. If I buy a apples at p pence a dozen and b apples at q pence a dozen and sell them all at one penny each, how much profit do I make?

56. A brick is a inches long, b inches broad, c inches thick; what is the length of all the edges added together?

57. A has 6 more marbles than B; B has x marbles. If A were to give one-third of his marbles to B, how many would B then have?

58. Find the number of pence in a sum of money made up of a guineas, b half-sovereigns, and c half-crowns.

59. If $x = a+b$ and $y = a-b$, find the value of $3x+4y-2a+3b$. Verify by putting $a = 2$, $b = 1$.

60. The prices of seats in the stalls, pit, and gallery of a theatre are respectively seven shillings and sixpence, half a crown, and one shilling. Find in sixpences the total value of x stall, y pit, and z gallery tickets. What is the value in pounds when $x = 40$, $y = 200$, $z = 300$?

CHAPTER II

THE USE OF SQUARED PAPER

11. It is often possible to illustrate an arithmetical or algebraical argument by means of squared paper. This is done by taking a side of a square to represent the unit

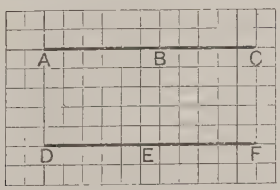


Fig. 1. Scale: side of small square represents 1 unit.

number; or it may be convenient to take some multiple or fraction of the side of a square to represent the unit. Every diagram should show clearly what scale is used.

Example I. *To show that* $6+5=5+6$.

Take a line AB to represent 6.

Produce it to C so that BC represents 5.

Then AC represents $6+5$.

Again, take DE to represent 5; produce it to F so that EF represents 6.

Then DF represents $5+6$.

It is seen that AC and DF are exactly the same length.

Hence the figure shows that $6+5=5+6$.

In a similar way the process of adding may always be represented, however many numbers have to be added.

Example II. *To show that* $17-9=8$.

Take AB to represent 17 units.

Measure BC from B towards A to represent 9 units.

Then from 17 units we have now taken 9 units.

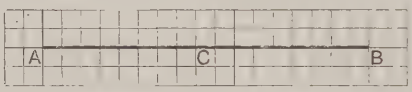


Fig. 2. Scale: side of small square represents 1 unit.

Hence AC represents $17-9$ units.

But by counting squares we see that AC represents 8 units.

Therefore $17-9=8$.

Example III. *To show that* $4\times3=12$.

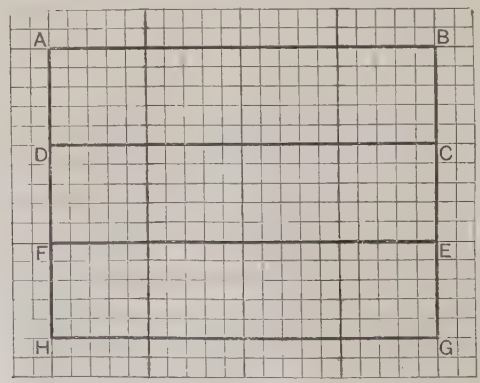


Fig. 3. Scale: 1 large square represents 1 unit.

It is sometimes found convenient to take the square itself to represent the unit.

In Fig. 3 1 large square represents 1 unit.

∴ 4 large squares represent 4.

i. e. the area *ABCD* represents 4.

So also do the areas *DCEF*, *FEGH*.

∴ the whole area *ABGH* represents $4+4+4$.

But 4×3 means $4+4+4$.

∴ area *ABGH* represents 4×3 units.

But the area *ABGH* is found by counting to contain 12 large squares.

Therefore Fig. 3 shows that $4 \times 3 = 12$.

Note. Any area such as *ABGH* is called a rectangle.

Example IV. *To show that the number of units of area in a rectangle is equal to the number of units of length in one side multiplied by the number of units of length in the adjacent side.*

We now have two different kinds of unit, the unit of length and the unit of area.

Take the side of one big square to represent the unit of length.

Then, in Fig. 3, the side *AB* is 4 units long and the side *AH* is 3 units long, so that the number of units of length in one side multiplied by the number of units in the adjacent side $= 4 \times 3 = 12$.

But if we take the square whose side is unit length to be the unit area, that is, if we take 1 large square to represent unit area, the rectangle is seen to contain 12 units of area.

Hence Fig. 3 shows, for a particular case, that the number of units of area in the rectangle is equal to the number of units of length in one side multiplied by the number of units of length in the adjacent side, provided the unit of area is the square whose side is unit length. The same argument could be applied in all cases when the sides of the rectangle contain an exact number of units.

Example V. *To show that* $x \times y = y \times x$ *when* x *and* y *are whole numbers.*

Draw a rectangle *ABCD* having *AB* x units long, *AD* y units long.

Then, as is shown in Ex. 4, the area of the rectangle is xy units of area.

Now suppose the rectangle turned round so that BC is in same direction as AB was before.

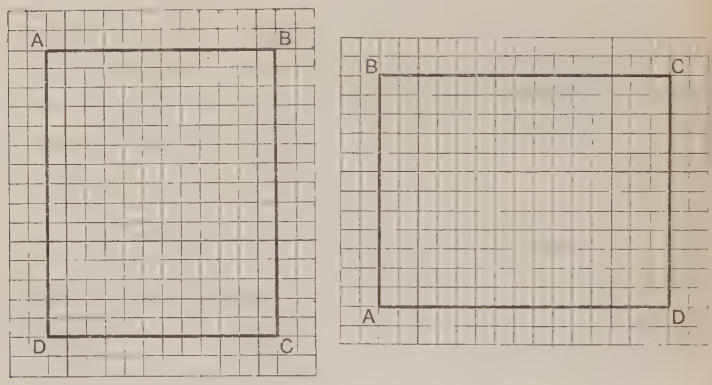


Fig. 4.

Then the area is now yx.

But since we are dealing with the same rectangle the area is the same.

Therefore $xy = yx$.

12. Powers and Indices.

Draw a square with side x units.

Then since both the adjacent sides are x units long, the area is xx units of area. Hence the product of two equal factors gives the area of a square; for this reason xx is called **the square of x or x square.**

Fig. 5.

Similarly, by means of cubes, each having the edges 1 unit long, it can be shown that the volume of a cube whose edge is x units long is xxx times the cube

with edge 1 unit. If the volume of the smaller cube is taken as the unit of volume, the volume of a cube whose edge is x units of length is xxx units of volume. Hence the product xxx is called the cube of x or x cube.

But we may have any number of equal factors multiplied together. If we start with 1, five successive multiplications by x give $xxxxx$.

The result of multiplying 1 by a number of equal factors is called a Power of that factor. If there are five equal factors, the product is called the fifth power, i. e. if one is multiplied five times by x the product is the fifth power. Instead of writing the product in full, we write the factor only once, but put a small figure at the top right-hand corner to show the number of equal factors, e. g. instead of $xxxxx$ we write x^5. This little figure is called **an Index.** The index 1, like coefficient 1, is always omitted. x is called the first power of x, x^2 means xx, and, as stated above, is called x square, x^3 is x cube, x^4 is read 'x to the fourth', and, similarly, **x^n means the product obtained by multiplying 1 by x n times, and is called x to the nth.**

13. An expression may contain different powers of the same letter, e. g. $3x^3+4x^2-2x+6$. This means that when we know the value of x, we must take three times the cube of that value, add four times the square, take away twice the first power of the value, and finally add 6.

When adding or subtracting, different powers are treated in the same way as different letters.

Example I. *Add together* $3x^3-4x^2+2x-1$, $2x^2-3x+4$, x^3+3x^2+4x-2.

$$\begin{aligned}\text{Sum} &= (3x^3-4x^2+2x-1)+(2x^2-3x+4)+(x^3+3x^2+4x-2)\\ &= 3x^3-4x^2+2x-1+2x^2-3x+4+x^3+3x^2+4x-2\\ &= 3x^3+x^3+3x^2+2x^2-4x^2+2x+4x-3x+4-1-2\\ &= 4x^3+x^2+3x+1.\end{aligned}$$

Or the work may be arranged in columns:

$$\begin{array}{r} 3x^3-4x^2+2x-1 \\ 2x^2-3x+4 \\ x^3+3x^2+4x-2 \\ \hline 4x^3+\ x^2+3x+1 \quad \text{Sum.} \end{array}$$

Examples II.

Squared paper is to be used for the Examples 1–10.

1. Show that $9+6-7=9-7+6$.
2. Show that $3(a+b)=3a+3b$.
3. Prove that $a+b=b+a$.
4. Try to find a meaning for $6-9$.
5. Show that $a(a+b)=a^2+ab$.
6. What does $(a+b)^2$ mean? Show that $(a+b)^2=a^2+2ab+b^2$. Verify this when $a=4$, $b=3$.
7. What is the value of $2-4+3$?
   What is the value of $4-2-3$?
8. Taking the side of a square to represent a mile, represent in a diagram the following places:

(i) A at the intersection of two lines; (ii) B 3 miles east of A and 2 miles north; (iii) C 4 miles east of A and 3 miles south; (iv) D 5 miles north of C and 6 miles west; (v) E 2 miles south of D and 3 miles west.

How far is A from C?

9. Show that $4(a-b)=4a-4b$.
10. Show that $(a+b)+(a-b)=2a$ and $(a+b)-(a-b)=2b$.

Find the value of each of the expressions from 11 to 15, when $a=1$, $a=0$, and $a=4$.

11. $3a^3-2a^2+2a+4$.
12. $a^4+2a^3-4a+18$.
13. $3a^3+2a^2-3a-2$.
14. a^5-4a^4+3a+2.
15. $2a-4a^3+3-2a^3+5a^2+a^4$.
16. Find the sum of $3x^2-4x+6$, $3x-2x^3+4$, $6x-4$.
17. What must be added to nothing to make $3x^2-x^3+1$?
18. From $4a^2+5ab+b^2$ take $3a^2+2ab$.
19. What is the sum of

$$2x^3+4x+5,\ 3x^2-x^3-2,\ 4+x+3x^2-x^2-3x+1\ ?$$

20. Simplify $x(x-2)-3(x^2-1)+4x(1+x)$.

21. Multiply x^2 by x^3.

22. Find the value of $3abc-2ab+4bc-3ac+2x^2$ when each of the letters equals x.

23. If $a=2$, $b=3$, find the value of

(i) a^2+ab+b^2; (iii) $3(2a+b)-2(2a-b)$;
(ii) $3a^2-4ab+2b^2$; (iv) $a^2b+2ab^2-3a^3+2b^3$.

24. Find the value of $y-(x-x^2-1)$ when $y=0$.

25. What must be taken from nothing to leave x^2-x+1?

26. A box is $2x$ inches long and $3y$ inches wide; what is the area of the top?

27. Find the area of a carpet whose length is x feet and breadth x inches.

28. A garden is x yards long and y yards wide. What is the cost of covering it with gravel at a shillings per square yard?

29. On a tour a man spent every day as many pence per mile as he walked miles. He walked x miles the first day, $2x$ the second day, and $3x$ the third day. How many shillings did he spend?

30. During a certain lesson each boy in a row obtained as many marks as there were boys in the row. There were 8 boys in the front row, 7 in the second row, and 4 in the third. How many marks were obtained altogether?

31. A square grass plot is surrounded by a gravel path 4 feet wide. If the side of the square is x feet, what is the area of the path? Illustrate your answer with a diagram, taking $x=20$.

32. How many cubic inches are there in a brick x inches long, y inches wide, y inches thick? How many such bricks are required to build a wall a yards long, b feet high, and c inches thick?

33. During each lesson on a certain day a master gave each boy for each answer as many marks as he had answers. If a boy answered twice in the first lesson, x times in the second lesson, and $2x$ times in the third lesson, how many marks had he?

CHAPTER III

NEGATIVE QUANTITIES

14. Example. *A boy has in his pocket* x *marbles, and* y *of them are lost through a hole in the pocket. How many has he left? What does the answer become when* (i) $x = 20$, $y = 4$; (ii) $x = 20$, $y = 20$; (iii) $x = 20$, $y = 25$?

Number of marbles left $= x - y$.

In case i, number left $= 20 - 4 = 16$.

In case ii, number left $= 20 - 20 = 0$.

In case iii, number left $= 20 - 25$.

Now it is impossible to take 25 from 20, so that we cannot find an answer in case iii.

Returning to the question, we see that if a boy has only 20 marbles it is impossible for 25 to drop through a hole in the pocket. Hence in this example we are not surprised to find that we cannot answer the question.

But some examples can be worked out in two different ways, and by comparing the two answers so obtained we can very often find a meaning for the result when a number is subtracted from a number smaller than itself.

15. Example. *A boy has* x *shillings in his pocket, but owes* y *shillings. What has he left after paying his debts? What does the answer become when* (i) $x = 5$, $y = 3$; (ii) $x = 5$, $y = 5$; (iii) $x = 5$, $y = 7$?

After paying debts he has left x shillings $- y$ shillings $= (x - y)$ shillings.

$\therefore$ in case i he has left $(5 - 3)$ shillings $= 2$ shillings.

in case ii he has left $(5 - 5)$ shillings $= 0$ shillings.

i. e. he has nothing left.

in case iii he has left $(5 - 7)$ shillings.

We might try to take 7 away in two steps, first take 5 away and then 2. This would give $5 - 7 = 5 - 5 - 2 = 0 - 2$.

So that the number of shillings left is the number left when 2 is taken from 0, and this cannot be done.

$0-2$ is two less than nothing; the zero is usually omitted, and we are left with the symbol -2 to represent two less than nothing.

But in case iii we really have the following question:—
A boy has 5 shillings in his pocket, but owes 7 shillings. What is left after using his money to pay his debts?

He still owes $(7-5)$ shillings.

i. e. he still has a debt of two shillings.

Comparing these two results, we see that having -2 shillings is the same thing as having a debt of 2 shillings.

Examples III a.

In the following set of examples each result should be worked for the three cases: (i) $x=5$, $y=3$; (ii) $x=5$, $y=5$; (iii) $x=5$, $y=7$. Try to find a meaning for all the answers.

1. A man stands with his back against a wall; he walks x yards forwards and then y yards back. How far is he now from the wall?

2. In the course of a certain day a merchant gains £x on some transactions, but loses £y on others. What is his gain on the day's work?

3. A man walks x miles due east from his house; he turns round and walks y miles in the opposite direction. How many miles east of his house is he now?

4. A labourer goes up x rungs of a ladder and comes down y rungs. How many rungs above the ground is he now?

5. Suppose that in example 4 the ladder rested in a hole with the tenth rung level with the ground; he now goes up x rungs from the ground and comes down y rungs. How many rungs above the ground is he now? How many rungs above the bottom of the hole is he?

Illustrate by means of a diagram on squared paper.

6. A cyclist starts from the twentieth milestone on the road from London to Brighton. He cycles x miles towards Brighton and then y miles towards London. How far is he now from his starting point? How far is he from London?

Illustrate by means of a diagram on squared paper.

7. In the Centigrade thermometer the freezing-point of water is 0° C. At 8 a.m. the reading is 0° C.; the thermometer rises x degrees, then falls y degrees. What is now the temperature?

8. A shopkeeper starts the day with £10 in his till; in the course of the day he takes £x from customers and pays bills to the amount of £y. What is his gain during the day? How much has he at the end of the day?

16. The student will have found that in several of the above examples a clear meaning can be attached to results which involve subtracting a number from nothing, and should now have little difficulty in understanding the following statements:

If we start with nothing and continually add on one, we obtain the ordinary whole numbers used in Arithmetic. These are called **Positive** numbers, and are sometimes distinguished by the + sign placed before them. As a rule the sign is omitted.

If we start with nothing and continually subtract one, we obtain another series of numbers which are called **Negative** numbers. These are distinguished by the sign -1; thus -5 denotes the negative number 5. The negative sign is never omitted.

17. Meaning of + and −. We now have two meanings for these signs. They may be used as verbs, in which case they give directions to add or subtract respectively; with this meaning they are **signs of operation.** They may also be used as adjectives, in which case they show whether the following number is positive or negative; with this meaning they are **signs of affection.**

When negative numbers are preceded by a symbol of operation they are enclosed in brackets; so too are positive numbers if the positive sign is inserted. Thus $5\times(-6)+(-3)\div(+2)-(-6)$ means multiply together the positive number 5 and the negative number 6, add the result obtained by dividing the negative number 3 by the positive number 2, and then subtract the negative number 6. In the next chapter we shall show how these operations are peformed.

18. There is another way of explaining negative numbers.

Suppose we start counting, and imagine the numbers arranged in a long straight line:

0. 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12.

If we wish to add 3 to 8, we start at 8 and then count forward three places, so obtaining 11.

If we wish to subtract 3 from 8, we start at 8 and then count three places backwards, so obtaining 5.

If we start with zero and count forward any number of places, say five, then we come to the number 5.

But what do we obtain if we start with nothing and count in the reverse direction? It has just been seen that counting three places backwards gives the same result as subtracting three; therefore counting three backwards from nothing will bring us to the number $0-3$, i. e. -3.

Hence we may say that Positive Numbers are obtained by counting forwards from nothing, and Negative Numbers are obtained by counting in the reverse direction.

Consider a lot of letters in a row:

				1	2	3	4	5	6
A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
−3	−2	−1	0						

Start counting at any letter and then continue counting. In the above line we have started at *E*, so that *J* is numbered 6. As we go to the right the numbers increase, but if we start at *J* and go to the left the numbers decrease one at a time; hence *D* should be numbered one less than 1, i. e. nothing or zero, *C* one less than zero, i. e. -1, and similarly $B-2$, $A-3$.

Graphic Representation of Negative Numbers.

19. By means of squared paper we can very simply obtain a geometrical representation of both positive and negative numbers. Such a representation is called a Graphic representation.

Take any point on one of the lines and call it *O*.

Then, as has been shown in § 11, any positive number, 9 for instance, may be represented by drawing a line from

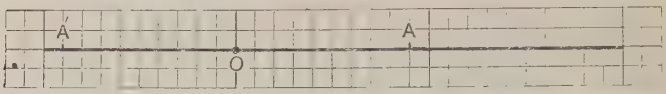


Fig. 6. Scale: side of 1 small square = 1 unit.

O to the right[1] equal to nine times the unit length. To do this we have to count nine squares to the right. The question arises, 'What will the line represent if we draw it to the left?' To do this we shall have to count in the reverse direction; but the last paragraph showed that by counting in the reverse direction we obtain negative numbers. Hence a line drawn to the left represents a negative number. In Fig. 6 OA represents $+9$, but OA' represents -9. Similarly if a line drawn upwards[1] represents a positive number, a line drawn downwards will represent a negative number, e.g. in Fig. 7 $OB = +9$, $OB' = -7$.

Fig. 7.

20. There is yet another way of regarding negative numbers. It has been mentioned in § 16 that any positive number can be obtained by continually adding one to nothing, i.e. $+5 = 0+1+1+1+1+1$; and that any negative number can be obtained by continually subtracting one from zero, i.e. $-5 = 0-1-1-1-1-1$.

Hence $+5+(-5) = 0+1+1+1+1+1$
$+0-1-1-1-1-1 = 0.$

[1] It is the *custom* to represent positive numbers by lines drawn towards the right or upwards; in such cases a line drawn to the left or downwards will, as is here shown, represent a negative number. If in any particular case it is more convenient to draw the positive lines to the left or downwards then lines drawn to the right or upwards will be negative.

Or $+x+(-x) = 0+1+1+1 \ldots$ the addition being repeated x times

$+0-1-1-1 \ldots$ the subtraction being repeated x times

$= 0.$

In words, this last result shows that any negative number when added to the positive number of the same size exactly cancels it, and the result is nothing. Thus a negative quantity is the exact opposite or reverse of the corresponding positive quantity, e. g. if $+6$ represents a distance measured to the right, -6 will represent an equal distance to the left.

Examples III.

1. If the rise in temperature during a certain period is represented by $x°$, what is meant when $x = -5$?

2. What is the result when the positive number 5 is added to the negative number 6?

3. By means of squared paper show that

$$5+(-6)+4 = 5+4+(-6).$$

4. Find the value of $3x$ when $x = 0$, when $x = +2$, and when $x = -3$.

5. A person A begins playing a game with x counters; he loses y, gains z, and then loses $2x$ counters. How many has he now? Find the value of the answer in each of the following cases:

(i) $x = 24,\ y = 6,\ z = 30$;
(ii) $x = 20,\ y = 3,\ z = 35$;
(iii) $x = 30,\ y = 10,\ z = 5$.

6. Find by squared paper or otherwise the value of $x+y$ when (i) $x = 0,\ y = -2$; (ii) $x = 3,\ y = -3$; (iii) $x = 1\frac{1}{2},\ y = 2\frac{1}{3}$; (iv) $x = 1\frac{1}{2},\ y = -2\frac{1}{3}$.

7. A boy possesses two shillings, but has been fined one shilling for breaking a window.

(i) If he is excused the fine, how much better off is he?

(ii) Some one gives him a shilling; how much better off is he?

8. What is left when y is subtracted from x? What must be added to y to make x?

Verify the results in the following cases: (i) $x = 5$, $y = 3$; (ii) $x = 5$, $y = -3$; (iii) $x = 5$, $y = 6$.

9. In the year 1908 a boy is x years old. In how many years will he be y years old? Verify by taking $x = 13$, $y = 17$ and $x = 13$, $y = 7$. Give the year in each case.

10. Find the value of xy in each of the following cases: (i) $x = +2$, $y = +3$; (ii) $x = -2$, $y = +3$; (iii) $x = +2$, $y = -3$; (iv) $x = -2$, $y = -3$.

11. What is the result of adding x to nothing nine times? What is the result of subtracting x from nothing nine times?

Explain the meaning of the negative quantity in each of the following cases:

12. A man gains -5 pounds.
13. An event happened -3 years ago.
14. The temperature fell -3 degrees.
15. A grocer lost -5 shillings every day.
16. The boy went up -5 places.
17. A man threw a stone up into the air a distance of -50 feet.
18. The match was lost by -5 runs.
19. The natives have -4 fingers on each hand.

CHAPTER IV

FURTHER CONSIDERATION OF THE SIMPLE RULES

Now that we have introduced negative numbers it is necessary to consider carefully what is meant by the simple operations Addition, Subtraction, Multiplication, and Division; and to find out whether it is possible to perform these operations with negative numbers.

Addition.

21. Consider the question in § 1, and suppose a child who could not add, but could count, were told to find out how many cows there were. He would go to the first field and

count up to 13 ; then to the second field and go on counting, the first cow in the second field being reckoned as 14, the next as 15, and so on up to 29. Similarly he would start at 30 in the third field, and finally find the total to be 34.

A little consideration will show that all simple addition sums in arithmetic can be done by counting, and in fact most beginners do addition sums by counting their fingers or strokes made on paper. Hence we see that addition is a short process to avoid a lot of counting.

It has been shown in § 18 that when we have a negative number we must count in the opposite direction to the direction we count when we have a positive number.

Suppose now we wish to find the value of $x+y$ in the following cases: (i) $x=+4, y=+3$; (ii) $x=+4, y=-3$; (iii) $x=-4, y=-3$; (iv) $x=-4, y=+3$.

Imagine the numbers arranged in a straight line as follows:

$-7.\ -6.\ -5.\ -4.\ -3.\ -2.\ -1.\ 0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.$

Case i. $x=+4,\ y=+3$.

Starting from 0 we count to the right to 4, then go on counting three more numbers, so obtaining 7 as the answer.

Case ii. $x=+4,\ y=-3$.

Starting from 0 we count to the right to 4, then count backwards three numbers, so obtaining 1 as the answer.

Case iii. $x=-4,\ y=-3$.

Here we start from 0 but count to the left to -4, then go on counting in the same direction, so obtaining -7 as the answer.

Case iv. $x=-4,\ y=+3$.

Again we start from 0 counting to the left to -4, but then count backwards to the right, so obtaining -1 as the answer

We have now shown that

$$+4+(+3)=+7$$
$$+4+(-3)=+1$$
$$-4+(-3)=-7$$
$$-4+(+3)=-1.$$

The student should go through the same series of additions with other numbers, as for example (i) $x=5,\ y=8$; (ii) $x=3,\ y=9$.

22. We now see that when a negative quantity is added the result is less than the original quantity, and that adding a negative quantity is the same thing as subtracting the corresponding positive quantity.*

Quantities which are all positive or all negative are said to be of **like sign**, quantities of which some are positive and some negative are said to be of **unlike sign**.

The general rules for addition are:

(1) If the quantities are of like sign add up the numbers and prefix + if the quantities are all positive, − if the quantities are all negative.

(2) If two quantities are of unlike sign, subtract that which is numerically smaller from the other, and prefix the sign of the greater.

Note.—Of the two numbers -8 and $+6$, -8 is really less than $+6$, but is said to be **numerically greater** than $+6$ because 8 is greater than 6.

(3) If there are any number of quantities, first add the positive quantities, then the negative quantities, and then obtain the final answer by Rule (2), e. g.

$$\begin{aligned} 5+(-2)+(+3)+(-7)+(+7) &= 5+3+7+(-2)+(-7) \\ &= +15+(-9) \\ &= +6. \end{aligned}$$

It should be noted here that we have assumed that additions may be performed in any order. This is obvious in the case of positive numbers (see § 11), and will be shown later to be true for negative numbers.

23. The same rules apply if, instead of mere numbers, we have quantities involving letters.

Example. *Find the sum of* $2a-3b+4c$, $3a+4b-2c$, $-5a+b-3c$.

Arrange the quantities so that the same letters come in the same vertical column.

* See note, page 20.

$$\begin{array}{r} 2a-3b+4c \\ 3a+4b-2c \\ -5a+\ b-3c \\ \hline +2b-\ c \end{array}$$ *Answer.*

[The mental work deals only with the coefficients.

$2+3=+5$ and -5 makes 0, $\therefore$ there are no a's.

$4+1=+5$ and -3 makes $+2$, put down $+2b$.

$-2+(-3)=-5$ and $+4$ makes -1, put down $-c$.]

The work could also be arranged as follows:

$$\begin{aligned} \text{Sum} &= 2a-3b+4c+3a+4b-2c-5a+b-3c \\ &= 2a+3a-5a-3b+4b+b+4c-2c-3c \\ &= 5a-5a-3b+5b+4c-5c \\ &= 2b-c. \end{aligned}$$

Subtraction.

24. Subtraction is the reverse operation to Addition, that is to say, it undoes the result of addition, e. g. if we add 7 to 8 we obtain 15; if we wish to reverse this we must subtract 7 from 15. Hence we expect that when we subtract we must do exactly the opposite to what we do when we add.

It has been shown in § 22 that in order to add a negative quantity we actually subtract the corresponding positive quantity; we therefore expect that in order to subtract a negative quantity we shall have to add the corresponding positive quantity. Let us see if this is reasonable.

Example I. *A boy has two shillings, and owes sixpence. How much has he really got? If he is excused the debt, how much has he?*

In the first case he really has $2s.-6d.=1s.\ 6d.$

In the second case he has $2s.$

So that taking away the debt has made him better off.

In symbols $1s.\ 6d.-(-6d.)=2s.=1s.\ 6d.+6d.$

Hence in this case subtracting a negative quantity has the same effect as adding the corresponding positive quantity.

Example. II. *A boy owes two other boys 6d. and 9d. respectively. How much has he? If he is excused the larger debt, how much has he?*

It has been shown that having a debt is represented by having a negative number of pence (or pounds or shillings).

Therefore in the first place the boy has $-6d.+(-9d.) = -15d.$

In the second place the boy has $-6d.$

That is $-15d.-(-9d.) = -6d. = -15d.+9d.$

Hence in this case subtracting a negative quantity has the same effect as adding the corresponding positive quantity.

In working the following examples it is useful to recollect that there are two ways of regarding subtraction. 15—8 may be taken to mean 'What is left when 8 is taken from 15?' This is the ordinary subtraction idea; or it may be taken to mean 'What must be added to 8 to make 15?' From this point of view subtraction is sometimes called **complementary addition.**

Examples IV a.

1. What must be added to -3 to make -7?
   What is the value of $x-y$ when $x = -7$, $y = -3$?

2. A man had £x ys. zd. in his pocket; he lost £3 zs. yd. How much was left? Express the answer in pence. Try to work this by ordinary compound subtraction, and so verify your answer.

3. At the beginning of the year 1907 a merchant's property was worth £3514; at the end of the year he owed £56. How much had he lost during the year?

4. At the beginning of the year a merchant had £x; during the year he gained £y. How much had he at the end of the year?
   Explain the question and the answer in the following cases:
   (i) $x = 500$, $y = -700$;
   (ii) $x = -700$, $y = 500$.

5. At noon on Sunday the temperature was 20° F., at noon on Monday it was x° F. What was the rise in temperature? What was the fall? What do the answers become when $x = 29$?

6. During the day the thermometer rises from x° C. to y° C. What is the rise in temperature? What does this become if $x = -5$, $y = 7$?

7. Prove in any way that $-6-(-5) = -1$.

8. In a 100 yards race between A and B, A had x yards start and was beaten by y yards. How far had A run when B was at the winning-post? Explain the question and answer when

$$x = -3,\ y = -5.$$

25. The student will now see that the suggestion of § 24 is correct, and we can state the **rule for subtraction** in the following form:

To subtract any quantity from another, change its sign and follow the rules for addition given in § 22, thus:

$$3x-(-2x) = 3x+2x = 5x.$$

This rule applies for single quantities or for quantities consisting of several terms.

Example. *From* $3a-4b+5c$ *take* $2a+2b-4c$.

We can adopt either of the two following modes of setting down the work.

$$\begin{array}{r} 3a-4b+5c \\ 2a+2b-4c \\ \hline a-6b+9c \quad \textit{Answer.} \end{array}$$

[Mental work. $3a-2a = a$, $-4b-2b = -6b$, $5c+4c = 9c$; the lower sign being changed in each case.]

Or Difference $= 3a-4b+5c-[2a+2b-4c]$

$$= 3a-4b+5c-2a-2b+4c$$
$$= 3a-2a-4b-2b+5c+4c$$
$$= a-6b+9c.$$

Notice that this agrees with what has already been shown in § 10 when dealing with the removal of brackets.

Graphical Illustration of Addition and Subtraction.

26. It has been shown previously that positive quantities may be represented by lines drawn to the right or upwards, and negative quantities by lines drawn to the left or downwards respectively.

To represent $+5+(-3)$ (Fig. 8).

Draw $OA = 5$ units to the right.

From A draw $AB = 3$ units to the left.

Then OB represents $5+(-3)$.

But $OB = +2$.

$\therefore\ +5+(-3) = +2$.

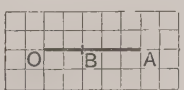


FIG. 8. Scale: side of 1 small square represents 1 unit.

To represent x + (−y) *when* x *and* y *are both positive and* x *greater than* y (Fig. 9).

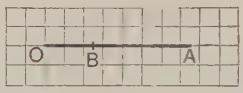


Fig. 9. Scale : side of 1 small square = 1 unit.

Draw OA to the right. As we do not know what x equals, we make OA any length; and as we are not told that x is a whole number, OA should not end at the end of a unit length.

Measure AB to the left; the length of AB is y. All that we know about y is that it is less than x, therefore B must be to the right of O.

Hence OB is positive and measures $x-y$ units.

$\therefore\ x+(-y) = +(x-y)$ when x and y are positive and x greater than y.

To represent x + (−y) *when* x *and* y *are both positive and* x *less than* y (Fig. 10).

Fig. 10.

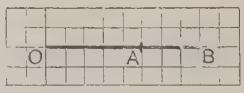


Fig. 11.

Scale : side of 1 small square = 1 unit.

Draw OA to the right to represent x units.

Measure AB to the left to represent y units. As y is greater than x, B is to the left of O, and $OB = y-x$ units in length, but is negative in sign.

$\therefore\ x+(-y) = -(y-x)$ when x and y are positive and x less than y.

In a similar manner the student can show that

$$(-x)+(-y) = -(x+y).$$

27. *To represent* $+5-(-3)$ *graphically* (Fig. 11).

Draw $OA = 5$ units to the right.

If we were to add (-3) units, we should now draw $AB = 3$ units to the left; but as we are to subtract, we draw AB in the reverse direction, i. e. $AB = 3$ units to the right.

Hence $OB = 5+3 = 8$ units to the right.

$\therefore\ +5-(-3) = +5+3 = 8.$

To represent x−(−y) *graphically* (Fig. 12) *when* x *and* y *are positive.*

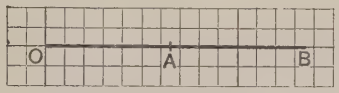


FIG. 12. Scale : side of 1 small square = 1 unit.

Draw OA to the right equal to x units.
Draw AB to the right equal to y units.
Then $OB = x+y$ units to the right.
$\therefore\ x-(-y) = x+y.$

Exercise. The student should show graphically that

$$-x-(-y) = -x+y,$$

and make two figures according as x is greater or less than y.

We have now shown graphically that the rules for addition and subtraction given in §§ 22 and 25 hold not only when the quantities are whole numbers but also when they are fractions.

Examples IV b.

1. Find the sum of $2a+3b-4c$, $3a-2b+c$, $-a+2b-c$, $2a-3b-5c$.
2. Subtract $3a-2b-c$ from $a+b+c$.
3. Simplify $3x^2-4x^3+7x-2+3x-6x+5-2x^3$.
4. If $a=5$, $b=3$, $c=2$, find the value of
   (i) $3a^2-2ab+bc-c^2+4bc-a^3$.
   (ii) $3a(b+c)-2b(4a-c)-c(b-c)$.
   (iii) $(a-b)+(b-c)+(c-a)$.
5. Add $3x^3-4x^2+1$, $3x^2+4x+x^3$, $3x+4+x^2$.
6. What must be added to $x+y+z$ to make $3x-4y-2z$? Verify by putting $x=1$, $y=z$, $z=0$.
7. Simplify $2ab-4a^2+2b^2-3ab+5a^2-2b^2+a^2-ab$.
8. From $2x^2-3x+1$ take $x+x^2-2$.
9. From $2a^2-3b^2+c^2$ take $3c^2+b^2-a^2$.
10. From $x^4-x^3+2x^2+3x-4$ take $2x^4+3x^3-2x^2+1$.
11. If a is greater than c, from £a bs. cd. take £c. bs. ad.
12. Find the sum of $4a^2-5ab+5ac-b^2-c^2$, $3a^2+2ab-c^2-2bc$, $-a^2-2ac+3b^2-5bc$, $b^2-2ac+c^2-3ab$.

13. From the sum of $3x^2-4x+1$ and $1-2x-3x^2$ take the sum of $2x^2-4x+3$ and $2x-1-x^2$.

14. Simplify $5+(-7)-(+5)+6-(-3)+4-(+8)$.

15. Add together $6a^3-(5a^2b-b^3)$, $7a^2b-(3ab^2-2b^3)$, $5ab^2-(4a^3+b^3)$, and subtract $2a^2b-2ab^2$ from the sum.

16. Add together $2a(a-2b-c)$, $b(4a-3b+5c)$, $c(2a-5b+c)$, and subtract a^2-4b^2 from the sum.

17. Find the sum of

$$3a^2-5a+2b,\quad 7a-4b-3b^2,\quad 2a^2-b+a-2b^2.$$

18. Find the value of $3x^2-5(x+7)-3(x^2-5x)-7$ when $x=4$.

19. When $a=2$, $b=0$, $c=-1$, find the value of

(i) $a(b+c)+b(c+a)$ and (ii) $(a+b-2c)^2-a^2+b^2-4c^2$.

20. Subtract $3x-y-2(x+y-z)$ from $3(x-y+z)$.

21. Add together $5a-(7b-c)$ and $3b-(9a+c)$, and subtract the sum from $c-4b$.

22. From $3x-(y-z)$ subtract $2z-(y-4x)$, and add $2x+y$ to the result.

23. If $a=2$, $b=3$, $c=5$, prove that a^2-ab+b^2 is greater than $bc+ca-c^2$.

24. What quantity must be subtracted from $-2x+3y-5z$ in order that the remainder may be $2x-7y-4z$?

25. Subtract the sum of $5a-\frac{5}{2}b+\frac{3}{2}c$ and $-\frac{3}{2}a+3b-\frac{1}{2}c$ from $\frac{5}{2}a+\frac{1}{2}b+c$.

26. To what expression must $10ab-22bc-14ac$ be added to produce zero? If $6x^2-14x+4$ be subtracted from zero, what will be the result?

27. What is the value of x when

(i) $x+2=6$; (ii) $x-2=-5$; (iii) $x+3=1$?

28. What number added to 16 will make -18? What must be added to -17 to make $+20$?

29. What must be subtracted from 16 to make 19?

Multiplication.

28. *Find the value of* xy *when* (i) $x=+4$, $y=+3$, (ii) $x=-4$, $y=+3$; (iii) $x=+4$, $y=-3$; (iv) $x=-4$, $y=-3$.

We will regard the second factor y as the **multiplier** and the first factor x as the number to be multiplied (the **multiplicand**, as it is called).

(i) $(+4)\times(+3)$ means $(+4)+(+4)+(+4)$ (§ 5)

$\therefore\ (+4)\times(+3) = 4+4+4$

$= 4\times 3$

$= 12.$

(ii) $(-4)\times(+3)$ means $(-4)+(-4)+(-4)$

$\therefore\ (-4)\times(+3) = -4-4-4$

$= -12$

$= -4\times 3.$

(iii) When the multiplier is negative, the interpretation of multiplication given in § 5 cannot be applied. We must therefore extend the meaning of multiplication, just as we have extended the meaning of addition and subtraction. Let us assume that when multiplying by a negative number we do exactly the reverse of what we do when we multiply by a positive number. Then instead of adding a certain number of times we shall subtract.

$(+4)\times(-3)$ will mean $-(+4)-(+4)-(+4)$

$\therefore\ (+4)\times(-3) = -4-4-4$

$= -12$

$= -4\times 3.$

(iv) $(-4)\times(-3)$ will mean $-(-4)-(-4)-(-4)$

$\therefore\ (-4)\times(-3) = 4+4+4$

$= +12$

$= +4\times 3.$

The argument would have been exactly the same if we had had any other whole numbers instead of 3 and 4.

Examples IV c.

1. Show that $(-5)\times(-6) = (-6)\times(-5)$.
2. When $a = 1$, $b = 2$, $c = 0$, $x = -1$, $y = -2$, find the value of
   - (i) $2a+3b-5x$.
   - (ii) $ab+bc+ca+xy$.
   - (iii) $3ax-4by$.
   - (iv) $a^2+b^2+c^2+x^2+y^2$.
   - (v) $3a^2b-2ab^2+3xy^2$.
   - (vi) $(a-b)\times(x-y)$.
   - (vii) $2ax+2cy-3bx+4ay$.
   - (viii) $3a^2x-2cy^2+3b^2x-4a^2y$.

(ix) $(a+b+c)-(x+y)$.

(x) $(ax+by)-x(ay-bx)$.

3. Show that $xy = yx$, whether x and y be positive or negative.

4. Find the value of $(2a+3b)(3x-4y)$ when $a=3$, $b=1$, $x=0$, $y=2$.

We can now state the rule.*

29. Rule for Multiplication. When two quantities are to be multiplied together, multiply the two numbers as in ordinary arithmetic, taking no notice of the signs, and then prefix + if the quantities are like (i. e. of same sign), — if the quantities are unlike (i. e. of different sign).

This is usually quoted in short as follows: 'Like signs give plus, unlike minus.'

It follows from this rule that a negative number multiplied by itself gives a positive product, so that the square of a negative number is positive, e. g.

$$(-5)^2 = +25, \quad (-\tfrac{1}{2})^2 = +\tfrac{1}{4}.$$

Division.

30. Just as in the case of subtraction, we may look at division in two ways. When dividing 12 by 3 we may regard ourselves as finding how many there would be in each part if 12 were divided into three equal parts, or as finding what number 3 must be multiplied by in order to give a product 12. This latter is the more general point of view, and gives us at once the rules for division.

We have $(+4)\times(+3) = 12$

$(+4)\times(-3) = -12$

$(-4)\times(-3) = +12$.

$\therefore\ +12 \div (+4) = +3$ [that is, +4 must be multiplied by +3 to give a product 12].

$(-12)\div(+4) = -3$

$(-12)\div(-3) = +4$

$(+12)\div(-3) = -4$.

The same argument applies to any numbers, and we have the rule.*

* See note, page 20.

Rule for Division. To divide one number by another, divide as in ordinary arithmetic, taking no notice of the signs, and prefix + if the quantities are like, — if they are unlike.

31. It has been shown that $a+b=b+a$, and $a\times b=b\times a$. Therefore we may talk of adding or multiplying two quantities. But $a-b\neq b-a$ * and $a\div b\neq b\div a$. Therefore **it is not correct to talk of subtracting two quantities or of dividing two quantities.** If we are to perform subtraction or division with two quantities, we must say '*subtract* a *from* b' or 'b *from* a', as the case may be; or '*divide* a *by* b' or 'b *by* a'.

Examples IV d.

1. Find the value of $xy+2xz-3x^2-2y^2+3yz$ when $x=2$, $y=-3$, $z=1$.

2. If $a=1$, $b=-2$, $c=3$, and $d=0$, find the numerical value of the expression $\dfrac{4a^2-b^2-2bc-cd}{3a-b(1-ac)}$.

3. If $a=1$, $b=-2$, $c=3$, and $d=0$, find the value of

$$\frac{b^2c^3}{a}+\frac{a^2c^2}{b}-\frac{b^2d^2}{c}.$$

4. When $a=6$, $b=5$, $c=-4$, find the numerical value of

$$2a(b^2-ac)-2b(b^2+ac)+\frac{2c-a}{a-b}.$$

5. Verify that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

when $a=2$, $b=\frac{1}{2}$, and $c=-1$.

6. Find the value of $\dfrac{-2x^2+3x+1}{x^2-7x+2}$

(i) when $x=-3$; (ii) when $x=0$.

7. When $a=-3$, $b=-2$, $c=4$, find the value of $(a-b+c)^3$ and of $2a^3-3a^2b+4a^2c$.

8. If $a=0$, $b=1$, $c=2$, $d=3$, find the values of

(i) $a(b-c)+d(a-b)+c(d-a)$;

(ii) $(a-b)(b-2c)(c-3d)(d-4a)$.

* The sign $\neq$ means is not equal to.

9. If $a = 3$, $b = 5$, $c = 0$, $d = -1$, find the numerical value of

$$\frac{(4a-3c)d}{3b-5d} + \frac{5abc}{9d} - \frac{2a-4d}{3b+c}.$$

10. If $a = 1$, $b = 3$, $c = -5$, and $d = 0$, find the value of $5abc - 8b^2c + 15ac^2 - 14acd$.

11. When $a = -2$, $b = -3$, $c = 4$, $d = 6$, $e = 0$, find the value of $a^2 + b^2 + c^2 - d^2 + e^2$ and of $3a^2 - 4ab^2 + 5cd^2 - 6abcde$.

12. What is the value of x when (i) $3x = -12$; (ii) $4x = 16$; (iii) $-5x = 20$; (iv) $-6x = -18$?

Examples IV.

1. Find the sum of $3x^3 - 2x^2 + 1$, $2x^2 - x^3 + x - 4$, $5x^3 - 2x + 3x^5$, $2 + 3x - x^2$.

Verify by putting $x = 10$ and $x = -1$.

2. A man walked 5 miles east from his home, then x miles west, and then 2 miles east. He was then 3 miles from home. Find the value of x.

3. What is the meaning of x^3? Show that $x^5 \times x^4 = x^9$. Find as shortly as possible the value of 7^6 and 2^{10}.

4. If $x = 0$, $y = 1$, $z = -2$, find the value of

(i) $2x - 3(2y - 3z)$;

(ii) $2xy + \frac{4z}{y} - 3yz^2$.

5. If x sheep cost £y, how many sheep will cost £z?

---

6. Find the value of

$$\frac{2ab}{cd} + \frac{3ac}{de} - \frac{4ad}{ce} + \frac{5(b+e)}{a+d}$$

when $a = -3$, $b = 0$, $c = 2$, $d = 4$, and $e = 6$.

7. A train runs a miles in x hours. Express its speed in feet per second.

8. Find the total cost (in shillings) of a drawing-boards at b pence each, a pencils at c shillings per gross, and a set squares at d pence per dozen.

9. State the rule of signs in Multiplication.

Multiply $4a - 3b + 5c$ by $-6x$.

10. A number x being taken, half of it is divided by 4 and the other half by 6. Express algebraically the sum of the two quotients; express also the difference between this sum and the quotient obtained by dividing the whole number x by 5. Supposing this difference to be 2, what was the given number?

---

11. Explain clearly the difference between 'index' and 'coefficient'. What are the rules for coefficients and indices in Multiplication (e. g. $3x^4 \times 4x^3$)?

12. Add together

$$-x^3-2ax^2+a^2x,\ 2x^3-ax^2,\ 4x^3-a^3,\ 5ax^2-a^2x-4a^3.$$

Test your answer by putting $x=2$, $a=-1$ in the four expressions and the sum.

13. If $a=1$, $b=-2$, $c=3$, $d=-4$, find the numerical value of

(i) $a^2+2ab-2acb^2-2bc+c^2$;

(ii) $\dfrac{a^2+b^2}{b+c}+\dfrac{b^2+c^2}{2a-d}+\dfrac{a^2+c^2}{b+d}$.

14. Multiply $3x^3-2x^2+x-1$ by $-2x^2$.

15. How many miles can a person walk in an hour if he walks a yards in 12 minutes?

---

16. To what expression must $3xy-22yz+13xz$ be added to produce zero? If $6p^2-11p-3$ be subtracted from zero, what will be the result?

17. If $a=0$, $b=1$, $c=-2$, and $d=4$, find the numerical value of

(i) $4bc-3b^2d+2abc+bcd$;

(ii) $\dfrac{b+c}{a^2+b^2}+\dfrac{2a+d}{b^2+c^2}+\dfrac{b+d}{a^2+c^2}$.

18. From $3ab+5cd-4abc-6bcd$ take $4cd+3ab-3abc-6bcd$.

19. B is x years old. Two years ago A was 5 times as old as B was 4 years ago. How old is A?

20. Is the following statement true when $x=3$:

$$14-x-21+11x-15=-5x-7+9x-14-3x+9?$$

---

21. If $y=3-2x$ and $z=3x-4y+2$, find the value of $3x-2y+3(z-2x)$ when $x=-2$.

22. What must $3a$ be multiplied by to give as the product $9a^2+6a-12a^3$?

23. A man gets x eggs for a certain sum of money, the price of each egg being y pence. How many would he get for the same sum if the price of each egg were $(y+z)$ pence?

24. Add together $3x^3-3ax^2+a^2x$, $-x^3+2ax^2-3a^2x+2a^3$, $3a^3-2x^3$, $2a^2x-3ax^2-2a^3$. Test the answer by putting $x=-1$, $a=-2$ in the four expressions and in the answer.

25. A boy had x shillings; he spent half his money on a bat, one-third on a ball, and had three shillings left. What was the value of x?

---

26. If $a=2$, $b=\frac{3}{2}$, $c=-5$, find the numerical value of $a^2+b^2-c^2-3ac+\frac{1}{4}ab+2bc$.

27. Distinguish between the meanings of (i) $(a+b)(a-b)$; (ii) $(a+b)a-b$; (iii) $a+b(a-b)$.

Find the sum of the three expressions.

28. How long will x men take to mow y acres of grass if each man mows z acres a day? How much longer would 3 fewer men take?

29. Is the following statement true when $y=-2$:

$$3y-2(y-6)+3=3(2-y)+5y+11?$$

30. Prove by means of squared paper that

$$(x+3)(x-2)=x^2+x-6.$$

---

31. Find the value of

$$\frac{(-8)(-9)}{(-6)(-3)}+\frac{(-9)(+10)(-12)}{(-15)(+14)}-\frac{(-20)(+18)}{-(6+4)(-9)}.$$

32. Which quantities are to be multiplied together in the following expressions?

(i) $a+b\times c+d$; (iii) $(a+b)\times c+d$,

(ii) $a+b\times(c+d)$; (iv) $(a+b)\times(c+d)$.

Find the value of each expression when $a=2$, $b=3$, $c=4$, $d=5$.

33. Write down (i) the number of yards of carpet x feet wide required to cover a floor y feet long and z feet broad; (ii) a proper fraction whose numerator and denominator are consecutive odd numbers of which the larger is $2n+1$.

34. If $2x=1$, $3y=-1$, $5z=1$, find the value of

$$\frac{1}{x+y}+\frac{1}{x+z}+\frac{1}{y+z}.$$

35. A snail crawls up a pole 6 feet high. If the snail crawls up x inches every alternate minute, and slips down y inches in each of the other minutes, how long will it take to reach the top of the pole? Verify by putting $x = 3$, $y = 2$. (Assume that the snail takes an exact number of minutes.)

---

36. Add together $3(2x+4y+z)$, $-2(x+y+z)$, $2x-(4y-5z)$, and find the value of the sum when $x = y^2 = 100z$ and $z = 1$.

37. Find the sum of $4x^2y-(3xy^2+2y^3)$, $2x^3-(x^2y-3xy^2)$, $x^3+4x^2y-3y^3$.

38. Show that $x^9 \div x^3 = x^6$. Divide $x^8-2x^7+3x^6$ by x^6.

39. Show by using squared paper that $x(x+y) = x^2+xy$. Also draw a figure for the case when $x = 5$, $x = -3$.

40. A person who has c shillings buys a things each costing b pence; how many pence will he have left? Explain the answer if $c = 5$, $a = 9$, $b = 8$.

---

41. Find the value of $2c^2+2a^2+2b^2-4bc+6abcd$ when $a = 1$, $b = -2$, $c = -3$, $d = 0$.

42. From $5a^2b-6ab^2-7a^3b^3$ take $10a^2b+8ab^2-8a^3b^3$.

43. Show in any way that $(-2)\times(-3) = +6$.

Multiply $6a^3bc-7ab^2c^2$ by $-5a^2b^2$.

44. A man has x sovereigns and y shillings. Express in symbols the facts that he has 30 coins and that their value is £11.

45. How long will it take a train to go 100 miles if it goes x miles in y minutes?

---

46. When $a = 3$, $b = -2$, find the value of

$$\text{(i)}\ \frac{3a-4b}{6a+5b}; \quad \text{(ii)}\ \tfrac{1}{4}(4a-5b)(4a+5b).$$

47. Subtract $\frac{3}{4}a-5b-\frac{3}{4}c+\frac{1}{3}d$ from $3a-\frac{7}{3}b+\frac{1}{6}d-\frac{1}{4}c$.

48. What is the meaning of Division? Give correct answers to the following: (i) $x \div 3$; (ii) $£x \div 3$; (iii) $£x \div £3$.

49. Fill up the gap in the following statement:

$$7+(6-2)-(3- \ ) = 9.$$

50. A sum of two pounds is divided among 3 boys, A, B, and C, so that A has x shillings, B has 6 shillings more than A, and C 5 shillings less than A. Express this algebraically. What is the value of x?

CHAPTER V

BRACKETS

BEFORE beginning this chapter the student should read once more §§ 9 and 10 in Chapter I. It is there shown how the need for the use of brackets arises, and how the brackets may be removed when no longer required. The following graphical illustrations of the use of brackets should be carefully studied:

32. *To prove that* $c(a+b) = ca+cb$ *when* a, b, c *are all positive.*

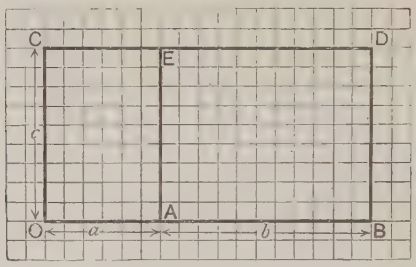


FIG. 13.

Draw OA equal to a units, and produce OA to B, making AB equal to b units.

Draw OC upwards equal to c units.

Complete the rectangle $OBDC$.

Now the length of OB is $a+b$.

$\therefore$ the area of the rectangle is $c(a+b)$. § 11. Ex. 4.

Mark off CE equal to a units, then $ED = b$ units.

Join AE.

Then area of rectangle $OAEC = ca$, and of rectangle $ABDE = cb$.

But the whole figure $OBDC$ = sum of $OAEC$ and $ABDE$.

$\therefore$ $c(a+b) = ca+cb$.

Note. No scale is given in Fig. 13. Suppose now that the side of a square represents $\frac{1}{12}$ unit, then the area of a square is $\frac{1}{144}$ sq. unit.

Then $OA = \frac{6}{12} = \frac{1}{2}$ unit, $AB = \frac{11}{12}$ unit, $OC = \frac{9}{12} = \frac{3}{4}$ unit.

$\therefore$ area of whole rectangle $OBDC = \frac{3}{4}(\frac{1}{2}+\frac{11}{12})$ square unit;

but area of $OAEC = 54$ squares $= \frac{54}{144}$ square unit,

area of $ABDE = \frac{90}{144}$ square unit.

$\therefore$ area of whole rectangle $OBDC = \frac{54}{144} + \frac{99}{144}$ square unit

$= \frac{3}{8} + \frac{33}{48}$ " "

$= \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{11}{12}$ " "

Hence $\frac{3}{4}(\frac{1}{2} + \frac{11}{12}) = \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{11}{12}$.

In a similar way it can be shown that $c(a+b) = ca+cb$ for any positive values, integral or fractional.

33. *To prove that* $c(a-b) = ca-cb$ *when* a, b, c *are all positive and* $a > b$.

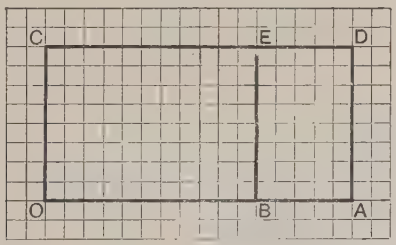


FIG. 14.

Draw a rectangle $OADC$ having $OA = a$ units, $OC = c$ units.

Measure AB backwards from A equal to b units and DE backwards from D equal to b units. Join BE.

Then OB equals $a-b$ units.

$\therefore$ area of rectangle $OBEC = c(a-b)$. § 11. Ex. 4.

Also area of rectangle $OADC = ca$ and of rectangle $BADE = cb$.

But rectangle $OBEC$ = rectangle $OADC$ − rectangle $BADE$.

$\therefore$ $c(a-b) = ca-cb$.

34. *To show that* $-c(a-b) = -ca+cb$ *when* a, b, c *are all positive and* $a > b$.

$-c(a-b)$ means subtract $c(a-b)$.

$\therefore$ $-c(a-b)$ means subtract $ca-cb$;

but in order to subtract we change the signs and add.

Hence $-c(a-b) = -ca+cb$.

35. *Find the sum of* $3x^2-2x+4$, ax^2+4x-c, $2x^2+bx+2$.

Arrange like powers in same column as directed in § 13.

$$\begin{array}{rrr} 3x^2 & - 2x & +4 \\ ax^2 & + 4x & -c \\ -2x^2 & + bx & +2 \\ \hline (1+a)x^2 & +(2+b)x & +(6-c) \end{array}$$

Answer.

[We have here followed the rule given in § 7. In the first column add up the coefficients of x^2, i.e. add $3+a-2$; this gives $1+a$. Therefore $(1+a)$ is the required coefficient of x^2. Hence we have the answer in the form shown.]

The above question might have been worded and worked in a different way, viz.

Arrange in descending powers of x

$$3x^2-2x+4+ax^2+4x-c-2x^2+bx+2.$$

$$\begin{aligned}\text{Expression} &= 3x^2-2x+4+ax^2+4x-c-2x^2+bx+2\\ &= 3x^2+ax^2-2x^2-2x+4x+bx+4-c+2\\ &= (3+a-2)x^2+(-2+4+b)x+(4-c+2)\\ &= (1+a)x^2+(2+b)x+(b-c).\end{aligned}$$

In working examples of this kind two points should be noticed:

(1) The first line of work should simply repeat the expression as in the question. Take care to copy the expression exactly as it is in the question. Many marks are lost in examinations through copying the question wrongly.

(2) The signs = should come one underneath the other, not in a slanting line.

36. Sometimes a straight line, called a vinculum, drawn above the terms, is used instead of a bracket. Thus $-\overline{7-a}x$ means the same as $-(7-a)x$.

The straight line of a fraction serves the same purpose as a bracket, e.g. $\dfrac{x-7}{3}$ is the same as $(x-7)\div 3$ or $\frac{1}{3}(x-7)$. It is very necessary to remember this when the minus sign is before the fraction.

Example. *Simplify* $8-\dfrac{x-3}{4}\times 12$.

$$\begin{aligned}\text{Expression} &= 8-\frac{x-3}{4}\times 12\\ &= 8-(x-3)\times 3 \quad \{\text{since 4 cancels}\}\\ &= 8-3x+9\\ &= 17-3x.\end{aligned}$$

Examples V a.

Simplify the following:

1. $3(5-4)-2(5-3)$.
2. $2\overline{7-x}+3\overline{2+5x}$.
3. $x(x-5)-x(x+4)$.
4. $12\times\frac{x-7}{3}-8\times\frac{x-3}{4}$.
5. $16(a+b-2c)+7(3a-b+2c)+8(2a-b-c)$.
6. $12(2x-y)-x(2-5)-y(7+3)$.
7. $\overline{a-b}\,x+\overline{b-c}\,x+\overline{c-a}\,x$.
8. $-p(q+r)-q(r+p)-r(p+q)$.
9. $(a-2b)c-(b-3a)c+(a+b)c$.
10. $(a-2b)c-(b-3c)a+(c+a)b$.

By graphic methods find the value of:

11. $(x+2)(x+3)$.
12. $(x-2)(x+3)$.
13. $(x+3)^2$.
14. $2x(a+b+c)$.
15. $(x+5)(x-5)$.

Arrange the following expressions in descending powers of x:

16. $3-2x+5-4x^2+2x^3-1$.
17. $2x^2-3x+4-x^2+7x^2+2x-5+x$.
18. $ax^2-3x+b-bx+cx^2+2x-5+ax$.
19. $p-qx+5x-px^2+qx^3-r$.
20. $(a+b)x^2+(b+c)x+(3-b)+(2-a)x+(2-b)x^2+a$.
21. Show graphically that $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$.
22. Simplify

$$6\times\frac{x-y}{2}+10\times\frac{2x-3y}{5}-12\times\frac{3x-5y}{6}.$$

Verify by putting $x=3$, $y=4$.

37. *Find the cost of* x *yards* y *feet of cloth at* x *shillings* y *pence per foot.*

Total number of feet $=3x+y$.

Each foot costs $12x+y$ pence.

$\therefore$ Total cost $=(3x+y)(12x+y)$ pence.

When two expressions in brackets are to be multiplied together the sign of multiplication is usually omitted, just as in the case of two single letters.

Now we could work the sum in a slightly different way.

One foot costs $12x+y$ pence.
$\therefore$ one yard costs $3(12x+y)$ pence.
And x yards cost $3x(12x+y)$ pence.
Also y feet cost $y(12x+y)$ pence.
$\therefore$ x yards y feet cost $3x(12x+y)+y(12x+y)$ pence.

Comparing this with the previous result, we see that

$$\begin{aligned}(3x+y)(12x+y) &= 3x(12x+y)+y(12x+y)\\ &= 36x^2+3xy+12xy+y^2\\ &= 36x^2+15xy+y^2.\end{aligned}$$

38. We have already shown that $c(a+b) = ca+cb$, where c may be any quantity we like.

Suppose that $c = x+y$, and substitute this value for c in the equality

$$c(a+b) = ca+cb.$$

We have $(x+y)(a+b) = (x+y)a+(x+y)b$
$$= ax+ay+bx+by.$$

Putting this into words, we have the rule:—**When two expressions in brackets are to be multiplied together, the product is obtained by multiplying every term in one bracket by every term in the other bracket, and taking the algebraic sum of the products.**

The above process can sometimes be reversed, that is, if we have an expression without any brackets, we may be able to put in or insert brackets.

For $ax+ay+bx+by = a(x+y)+b(x+y)$
$$= (x+y)(a+b).$$

Examples. $3x+4y+3a+4b = 3x+3a+4y+4b$
$$= 3(x+a)+4(y+b).$$
$$3x+6y+x^2+2xy = 3(x+2y)+x(x+2y)$$
$$= (x+2y)(3+x)$$
$$= (x+2y)(x+3).$$

Examples V b.

Remove brackets in the following examples:

1. $(x+2)x$.
2. $3(x+2)$.
3. $(x+2)(x+3)$.
4. $(x-2)x$.
5. $3(x-2)$.
6. $(x-2)(x-3)$.

7. $(x+5)(x+6)$. 8. $(x-3)(x+4)$. 9. $(x-a)(x-2a)$.
10. $(a-4)(a+2)$. 11. $(a+3)(b+2)$. 12. $(p-x)(2p+3x)$.

By graphical methods remove the brackets from

13. $(x+1)(x+2)$. 14. $(x-3)(x+2)$. 15. $(x-5)^2$.

Insert brackets in the following expressions:

16. x^2+3x. 17. $2x+6$. 18. $x^2+7x+12$.
19. x^2-3x. 20. $2x-6$. 21. $x^2-7x+12$.
22. x^2-ax. 23. $bx-ab$. 24. $x^2-ax-bx+ab$.
25. $ax-ab+ac$. 26. $a^2-ab-ac+bc$.

What expressions multiplied together give the following products?

27. $5x-10$. 28. $xy-2y-3x+6$. 29. $x^2+8x+16$.

30. A farmer leaves to each of his x sons 5 fields, and to each of his y daughters 3 fields. Each field contains $2x$ acres y roods. What was the area of the whole farm?

31. There are x houses on one side of a street, and y houses on the other. How many houses are there in the street? The rent of each house is £x ys., what is the total rental value of the street in shillings?

32. What is the area of a rectangle $(x+4)$ feet long, $(x-2)$ feet wide?

33. Each square of a chess-board is 1 square inch in area; the border is x inches wide. What is the area of the whole board?

34. A man sells x oxen and y sheep for $(2ax+2ay)$ pounds; what was the average price received for each animal?

35. A gives B £a, B gives C £b, C gives A £c, and then each has £x. How much had each at first?

39. A bracket may in several respects be compared to a parcel. Just as a person would rather do up half a dozen objects into one parcel than carry them separately, so in Algebra quantities are placed in a bracket when it is more convenient to deal with them as one quantity than to deal with them separately.

Again, the person would not undo the parcel unless he wanted something inside the parcel; similarly, in Algebra, a bracket should not be removed unless some term inside the bracket is particularly wanted. Usually terms containing x are wanted.

Example I *Arrange in ascending powers of* x *the following expression,* $(ax+b)\,c+(a+b)\,x+bx\,(x^2+cx)$.

Expression $= (ax+b)\,c+(a+b)\,x+bx\,(x^2+cx)$.
$= acx+bc+(a+b)\,x+bcx^2+bx^3$
$= bc+x\,(ac+a+b)+bcx^2+bx^3$.

Here the first and third brackets are removed because x is inside them, but the second is kept because x is not inside it.

Example II. $(x+a)(x+b) = (x+a)\,x+(x+a)\,b$
$= x^2+ax+bx+ab$
$= x^2+(a+b)\,x+ab$.

Here the brackets are first removed because x is inside them, and then a bracket is inserted in order to get one term $(a+b)\,x$ instead of two terms $ax+bx$.

40. A parcel may contain another parcel, the second parcel may contain a third parcel, and so on. In the same way a bracket may contain other brackets. When this is the case brackets of different shapes are used to avoid confusion.

Example. *A certain number* x *is subtracted from* 10, *the remainder is subtracted from* 7. *This second remainder is multiplied by* 5 *and the product added to the original number. This last result is squared; find an expression for the final result.*

First remainder $(10-x)$.
Second remainder $\{7-(10-x)\}$.
Multiply by 5 $5\{7-(10-x)\}$.
Add to x $[x+5\{7-(10-x)\}]$.
Square $[x+5\{7-(10-x)\}]^2$.

It is not often that more than the three kinds of brackets here shown are wanted. If they should be wanted, the same shapes are used again, but made larger. The necessity for more brackets is often avoided by using a vinculum.

When brackets are to be removed, it is usually more convenient to remove the innermost bracket first, and to work outwards, but it is quite possible to remove the outermost first and to work inwards.

The brackets in the above question could be removed in either of the following ways:

(i) Working from within

$$[x+5\{7-(10-x)\}] = [x+5\{7-10+x\}]$$

removing inside bracket,

$$= [x+5\{-3+x\}]$$

collecting,

$$= [x-15+5x]$$

$$= 6x-15;$$

$$\therefore [x+5\{7-(10-x)\}]^2 = (6x-15)^2$$

$$= (6x-15)\,(6x-15)$$

$$= (6x-15)\,6x-15\,(6x-15)$$

$$= 36x^2-90x-90x+225$$

$$= 36x^2-180x+225.$$

(ii) Working from without

$$[x+5\{7-(10-x)\}] = x+5\{7-(10-x)\}$$

$$= x+35-5(10-x)$$

$$= x+35-50+5x$$

$$= 6x-15;$$

and the work would be finished as in method (i)

Examples V c.

Remove the brackets and simplify:

1. $12a-[6a-2\{3a-4(b-a)\}-(9a+8b)]$.
2. $3a-[2b-\{4c-12a-(4b-8c)\}-(6b-12c)]$.
3. $3a-[5b-\{6a-(4a-7b)\}]$.
4. $x-\{2(y-z)+3z\}$.
5. $4(x+y)+2[x+3\{y-2z-(2x+y)\}]$.
6. $-[2x-\{y-z-(3x-y)\}]$.
7. $-[2y-\{z+y-(3z+2x)\}]$.
8. $6a-2[b-4(3c-2x)+3\{a-(4c+x)\}]$.
9. $3p-[q-3\{p+2(q-p)\}-(3q+p)]$.
10. $[2a+b-\{4a-(5a-\overline{a-2b-c})\}]+[a-\{a-(a-\overline{a-b})\}]$.
11. If $a=-10$, $b=8$, $c=1$, find the value of

$$2a-3b+c-[-3a-\{2c-4b+(5b-a)\}] - [4a+\{b-c-(2c+b-a)\}]$$

12. If $a=-3$, $b=2$, $c=0$, find the value of

$$2b-a[-a+\{b-2c(b-a)\}+b].$$

13. Simplify $6\left\{\frac{x-7}{3}-2\left(x+\frac{x+4}{12}\right)-x\right\}$.
14. Simplify $2a-\{3a-(4b+2a)\}+\{5a-(4b-a)\}$.

15. If $a = 3$, $b = 1$, find the values of

(i) $2a^2-(3ab-b^2)$; (ii) $2a^2-3(ab-b^2)$; (iii) $2\{a^2-3(ab-b^2)\}$.

16. Find the value of

$$[a-2\{b-(c-d)\}]\,[3a-\{b+2(c+d)\}] - [a-b+3(c-d)]\,[3a+b-2(c-2d)],$$

when $a=1$, $b=2$, $c=-1$, $d=-3$.

17. Simplify $2x-[y-2\{x+(y+x)+2y\}+2x]$.

18. Simplify $3\{(3a-b)^2+(a-3b)^2\}-10(3a-b)(a-3b)$ and verify the result when $a=2$, $b=1$.

19. Simplify $3(a+c)-5(a+b+c)-[b-\{c+2a-(b-c)\}]$.

20. Find the sum of $[-5x+7y-\{5x-(6x+7)-(4y+1)\}]$ and $-[-(4x+7y)+\{8-(-8x+9y)+(5x-8y)\}]$.

Examples V.

1. Find the value of $a+b-[a-\{c+2a+b+(2c+b)\}]$ when $a=3$, $b=4$, $c=2$.

2. Subtract $2x-3(y+2z)$ from $3(x+y)-8z$.

3. Add together $a-\{b+(c-2d)\}$, $2(3a+b)-4(c+d)$, and $5d+3\{c-(2a+b)\}$.

4. Subtract $q-[q-\{q-(q-\overline{q-p})\}]$

from $p-[p-\{p-(p-\overline{p-q})\}]$.

5. Add together $a-\{b+c-(a+b)\}+c$, $2(3a+2b)-4(b+2a)-c$, and $3(2b-a)-2(3b-a)+c$.

6. Find the remainder when $3a-5\{a-b-(c-2b)\}+b$ is taken from $c-2[c-\{a-2(c-b)\}]$.

7. Subtract $(a-b)(b+c)$ from $(b-c)(c+a)$.

8. Simplify $x(2y-3z)-2y(z-3x)+3z(x-2y)$.

9. Simplify $-7[-11x-4\{-17x+3(5x-1)\}-3(x+4)]$.

10. Arrange $a(b^2-c)+b(c^2-a)+c(a^2-b)$ in descending powers of c; and find the value when $a=2c$, $b=3c$.

11. Simplify $7x^3-(3-4ax)-3x(4x+1)(ax+5)$; and re-bracket the result according to descending powers of x.

12. What is the value of $a(x^3-ay^3)+xy(x-a^3y)$ when $x=ay$?

13. A dealer buys p horses for £a each; he sells q of them for £b each, and the rest for £c each. Find an expression for his gain on the transaction.

14. A man's salary is £x a day and he spends £y a day. How much does he save in z days?

15. A house contains p rooms on the ground floor and q rooms on the first floor. In each room there are p pictures on each

short wall and q on one of the long walls; how many pictures are there altogether?

16. In a certain field there are x cows and y sheep; the average value of each animal is x pounds y shillings. Find the total value of the animals.

17. A big box contained 50 apples and a large basket. In the basket there were 38 apples, a paper bag, and a cardboard box. The bag contained 6 apples; in the cardboard box there were 5 apples and a bag containing 3 apples. How many apples were there altogether?

18. If $y = 2x - 3$, express $3x + 5y$ in terms of x only.

19. If $y = 3x + 2$, what are the values of y when $x = 0, 1, 2, 3$ respectively? What value of x will make $y = 14$? Is there any value of x which makes y equal to x?

20. If $y = 3x + 2$ and $z = 3y - 4$, express z in terms of x. Find the value of z when $x = -3$ by two methods.

CHAPTER VI

EASY FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION

41. If a quantity continually changes its value it is called a **variable**; a quantity which does not change its value is called a **constant**. x, y, z are usually used to denote variable quantities; a, b, c are used for constant quantities. If a man is running, the time he has been running and the distance he has run are continually changing—they are variables; but his speed may be the same throughout; in such a case speed would be a constant.

42. Very often two variables are so related to one another that if one is known the other can be found.

Example. *A man spent £x in buying* y *cricket balls at* 5 *shillings each.*

Here $20x$ = number of shillings spent
and $5y$ = number of shillings spent.
$\therefore\ 5y = 20x.$
Divide by 5. $y = 4x.$

Hence if x is known, y can be found. In such cases as this x is called the independent variable and y the dependent variable.

43. Example. *If* $y = 3x^2 - 2x + 5$, *make a table showing the values of* y *corresponding to the integral values of* x *between* -3 *and* $+3$.

If $x =$	-3	-2	-1	0	1	2	3	
then $x^2 =$	9	4	1	0	1	4	9	
$3x^2 =$	27	12	3	0	3	12	27	
$-2x =$	6	4	2	0	-2	-4	-6	
and $y =$	38	21	10	5	6	13	26	

Any expression whose value depends on the value of x is called a function of x. Thus $4x$, $3x^2 - 2x + 5$, $3 + \frac{2}{x}$ are all functions of x.

Examples VI a.

Make tables showing the values of the following functions as x changes from -2 to $+5$.

1. $y = 3x + 2$.
2. $y = 2x - 4$.
3. $y = 3x^2 - 2x + 1$.
4. $y = 2x^2 + 4x - 5$.
5. $y = x^2 - 4x + 3$.
6. $y = x^3 - 2x$
7. $y = 3 + 2x$.
8. $y = 3 + 2x^3$.
9. $y = 4 - x$.
10. $y = x - x^3 + 2$.

GRAPHICAL REPRESENTATION OF FUNCTIONS

44. It has been shown in § 19 that there are two ways of representing a quantity graphically; lines may be drawn horizontally, i. e. to the right or left, or vertically, i. e. up or down. By using both methods we can represent two variables in the same figure.

Example. *To show in a figure that when* $x = 4$, $y = 7$.

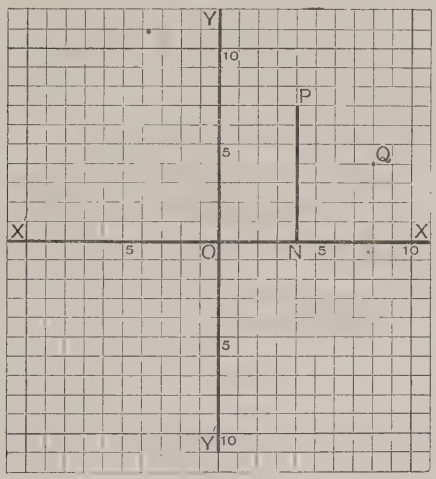


Fig. 15.

Using squared paper take any point O where two lines cross; see Fig. 15.

Measure $ON = 4$ to the right, then ON represents $+4$.

From N measure NP upwards, then NP represents $+7$.

It is always understood that horizontal lines represent x and vertical lines represent y; hence any one looking at Fig. 15 can see at once that when $x = 4$, $y = 7$.

The line NP is not always actually drawn; it is sufficient to count the squares and to make a dot or small cross when the required numbers have been counted. Thus in Fig. 15 the point Q shows that when $x = 8$, $y = 4$.

When values of x and y are represented in this manner they are said to be plotted.

45. The point O is called the 'origin', because it is the point from which the measurements start. When the point O has been chosen, the horizontal and vertical lines are drawn to show in what directions the measurements are to be made. These lines are called the axes; the horizontal line is the x-axis and is named $X'OX$, the vertical line is the y-axis and is named YOY'.

The next step is to choose the scale. When fractional values do not occur it is usually convenient to take the side of 1 small square to represent 1 unit.

Having chosen the unit, mark off every fifth unit along the axes. Except when the side of a square represents the unit, the scale should be written in one of the corners of the figure, as has been done in previous sections.

Recollect that negative values of x are measured to the left and negative values of y are measured downwards.

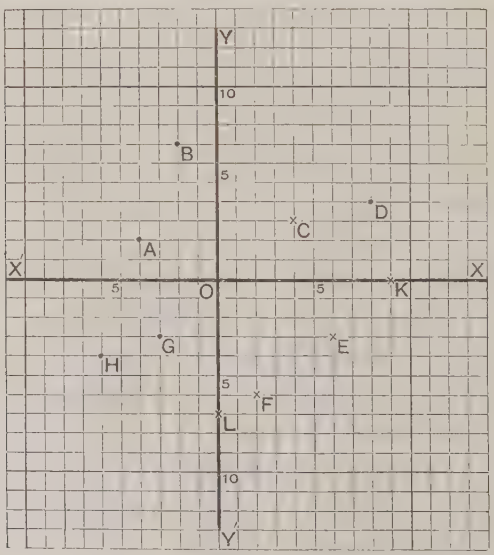


Fig. 16.

Examples VI b.

1. Show graphically in one figure that:

when $x =$	3	4	5	-1	-2	-7	-9	6
then $y =$	-4	-7	2	$+3$	-6	$+3$	-9	5.

2. What relations between x and y are represented by the points A, B, C, D, E, F, G, H, K, L in Fig. 16?

3. Represent in a diagram the values of x and y given in the table of § 43.

46. The graphical method described in the last section is often used to show how quantities change from time to time. A very common example is the diagram that appears in newspapers to show the height of the barometer at different times during the day. In making these diagrams, the lines that are to represent time are generally measured along the x-axis, the other quantity being represented by lines perpendicular to the x-axis.

Example. *During a certain week the temperatures at noon are given by the following table:*

	Sun.	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*
Temp.	29°	35°	42°	31°	27°	43°	50°

Represent this graphically.

1. As there are no negative quantities, measurements may start from a point near the left-hand bottom corner instead of from a point in the middle of the figure, see Fig. 17.

2. As no temperature is less than 27°, it will save space if we begin counting the temperatures at 25°, i. e. 27° will be 2 units up, not 27.

3. In order not to cramp the figure, take twice the side of a square to represent a day.

4. When all the points have been put in, join up the consecutive points by straight lines.

The broken line formed by joining the points is called a **graph.**

The graph is shown in Fig. 17; it tells us all that the table of the question does, and also shows at a glance which day was the coldest at noon, which day had the lowest

average temperature, which day had the greatest change of temperature, &c. But this graph does not enable us to find the temperature at any other times than noon on the respective days.

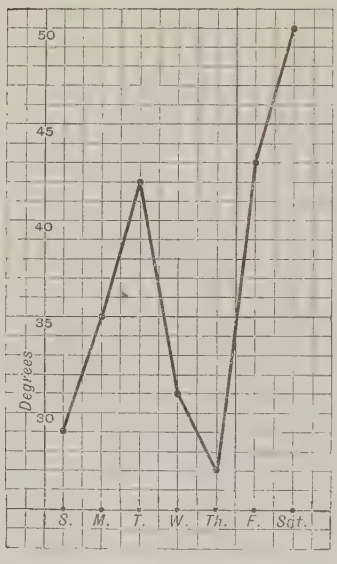


Fig. 17.

Scale: side of square = 1°.

If, however, the temperatures had been taken every half-hour the points on the diagram would have been much nearer together, and if the same scale had been used the points might be joined by a curved line instead of by a series of straight lines. In such a case the graph would give a very good idea of what the temperature was between the readings, as well as what it was at the actual times of the readings.

Examples VI c.

Draw graphs from the following tables:

1. A boy's position in form is given by the following table:

Date	Jan. 30.	Feb. 6.	Feb. 13.	Feb. 20.	Feb. 27.	Mar. 6.
Place	5	12	17	6		15

Draw a graph. Can you draw any conclusion as to his place on February 27?

2. In consecutive cricket matches a boy made the following scores: 17, 4, 0, 6, 3, 12, 1, 2, ?, 7, 3.

Draw a graph. Can you draw any conclusion as to the missing score?

3. A thermometer was placed in some water which was then heated, and the temperature was read every minute. Draw a graph from the following table:

Temperature in degrees	12	16	19·5	22·5	25	26·5	27
Time in minutes	0	1	2	3	4	5	6

Would this graph give any information as to the temperature after $4\frac{1}{2}$ minutes?

4. If $y = 3x - 2$, calculate the values of y when x has the integral values from -5 to $+5$; and draw the graph.

From the graph try to find the values of y when $x = -\cdot5$ and $x = +\cdot25$. Check the answers by actual calculation.

5. A man sells kettles. He has only made them of three sizes as yet, and he has fixed on the following as fair list prices:

Capacity in pints	12	6	2
Price in pence	68	50	22

He knows that other sizes will be wanted, and he wishes to publish at once a price list for many sizes. By drawing a graph obtain approximate prices for 4- and 8-pint kettles.

6. The keeper of a restaurant finds that when he has G guests a day his total daily profit is P pounds. The following numbers are averages obtained by comparing many days' accounts. Draw a graph, and so obtain approximately the number of guests for which there would be neither gain nor loss.

G	210	270	320	360
P	−0·9	+1·8	+4·8	+6·4

7. A train leaves Manchester at 6 a.m.; the following table shows the distance it has travelled at certain times. Draw a graph, and find (i) the approximate time it takes to travel 9 miles, (ii) how far it has travelled in 40 mins.

Distance in miles	1	4	5	10	12	16	18
Time	6.4	6.11	6.16	6.28	6.36	6.47	6.52

8. The following table shows the time for lighting cycle lamps on certain days in a certain month:

Date	1	2	5	8	11	13	17	20	24	28
Time	5.46	5.48	5·53	5.57	6.3	6.7	6.15	6.20	6.28	6.35

Draw a graph, and find the lighting-up time on the 4th, 15th, and 22nd.

9. The height of the barometer at noon, on certain days in June, is given in the following table:

Height	30·13	30·10	30·10	30·23	29·96	29·83	29·43
Date	1	2	3	4	5	7	8

Draw a graph, and if possible find the height on the 6th.

10. The amateur records for certain distances are given in the following table. Represent them in a diagram.

Distance in yds.	100	120	150	200	250	300	440
Time in secs.	$9\frac{3}{5}$	$11\frac{4}{5}$	$14\frac{4}{5}$	$19\frac{1}{5}$	$24\frac{1}{5}$	$30\frac{3}{5}$	$47\frac{4}{5}$

11. Represent in a diagram the winner's time in the University Boat Race as given in the table below.

Year	1890	1891	1892	1893	1894	1895	1896
	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.
Time	22 3	21 48	19 21	18 47	21 39	20 50	20 2

12. The following table shows the number N of articles that may be bought for £1 when the price is x pence:

x	5	6	8	10	12	20	24	30	40	48
N	48	40	30	24	20	12	10	8	6	5

Draw a graph. How does this graph differ from the previous graphs?

13. In a price list the following prices are given for tree pruners:

Length in feet	2	4	6	8	10
Price	3*s.* 6*d.*	4*s.* 3*d.*	5*s.*		6*s.* 6*d.*

Represent these prices by a graph, and try to find the price of a pruner 8 feet long.

14. The price of dish covers is given in the following table. By means of a graph find the approximate price of a 12-inch cover.

Length in inches	10	12	14	16	18
Price	9*s.* 3*d.*		15*s.* 9*d.*	20*s.*	26*s.* 3*d.*

47. Graphs of algebraical functions.

Example. *Draw the graph of* y *when* $y = x^2 - x$.

Step 1. Tabulate the values of y as in § 43.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	
x^2	25	16	9	4	1	0	1	4	9	16	25	36	
y	30	20	12	6	2	0	0	2	6	12	20	30	

Step 2. Choose the scale. This depends to a great extent upon the size of the piece of squared paper that is to be used. If we wish to graph all the above values, the paper must be big

enough to show $y = 30$; as there are no negative values of y, about 35 squares in length will allow us to take the side of 1 square to represent 1 unit. In Figs. 18 and 19, the points that

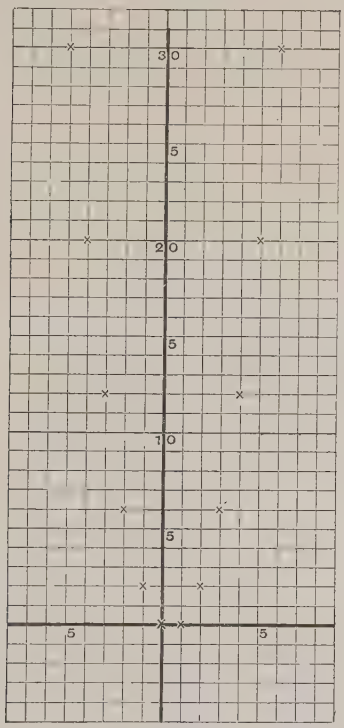


Fig. 18.
Scale: side of 1 square = 1 unit.

represent the values for x and y obtained in Step 1, are shown by crosses. Sometimes it is necessary to take different scales for x and y. In this example, if the squared paper were a small piece, it might be advisable to take the side of a square to represent two units for y, but only half a unit for x. This is the scale used in Fig. 19.

Step 3. Should the graph be finished by joining consecutive points by straight lines, or ought a curve to be drawn passing

through all the points? A little consideration shows that a slight alteration in x will produce only a slight alteration in the value of y. In all such cases a curve should be drawn, provided sufficient points have been obtained to make the shape of the curve quite clear. In Fig. 19 the points are closer than in Fig. 18, and the form of the curve is more obvious. In both curves there is no indication

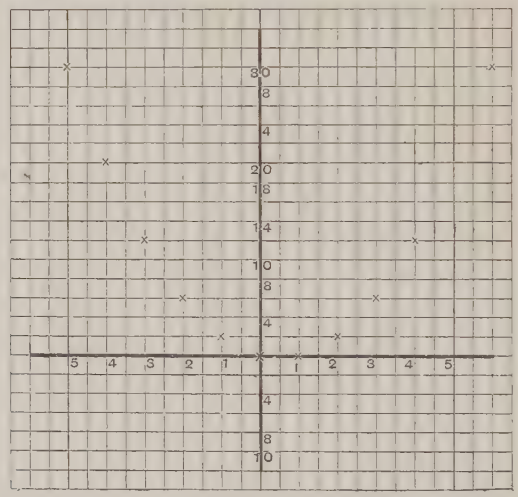


Fig. 19. Scale: 2 sides = 1 for x; 1 side = 2 for y.

of the shape of the curve between $x = 0$ and $x = 1$. We must therefore find the values of y corresponding to values of x between 0 and 1. With a small scale it is no use working the values of y correct to more than two decimal places.

x	·2	·4	·5	·6	·8
x^3	·04	·16	·25	·36	·64
y	−·16	−·24	−·25	−·24	−·16

It is now seen that the least value of y is $-\frac{1}{4}$, and that this corresponds to $x = \frac{1}{2}$. This point must be found as accurately as possible, the sides of the square being divided by eye measurement.

Step 4. The graph should be finished by drawing a curve through all the points obtained. It should be a graceful curve without any awkward bends.

48. It often happens that only a portion of a graph is actually wanted; when this is the case, a larger scale can be used. Suppose the graph of $y = x^2 - x$ were required only from $x = 0$ to $x = 1$. The values of x and y have been obtained in the last section. We can now take the side of a square to represent only one-tenth of a unit. The resulting graph is shown in Fig. 20.

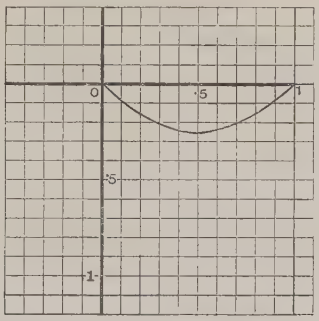


FIG. 20. Scale: 1 side = $\frac{1}{10}$ unit.

Examples VI d.

Draw the graphs of the following functions, and in each case find the value of y when $x = 3{\cdot}5$ and verify by calculation:

1. $y = x$.
2. $y = x^2$.
3. $y = x^2 + x$.
4. $y = x + 2$.
5. $y = 2x + 3$
6. $y = 3x^2 - 4$.
7. $y = 3 - 4x$.
8. $y = \frac{x^3}{4}$.
9. $y = 2x^2 - 5x + 6$.
10. $y = 7x - 9$.

In the following graphs find the value of x when $y = 2{\cdot}7$ and verify by calculation:

11. $x = 2y + 3$.
12. $x = 3y - 5$.
13. $x = y^2$.
14. $x = 3y^2$.
15. $x = 3y^2 + y$.
16. $x = y^2 - 2y + 4$.
17. $x = 3 - y^2$.
18. $x = 4 - 2y$.
19. $3x - 4y = 1$.
20. $4x + 5y = 20$.

49. Some of the graphs in Examples VI d are curves,* some are straight lines. It should be noticed that when the function is of the first degree (i. e. containing only the first power of the independent variable and no fractions with the variable in the denominator) the graph is a straight line. This is found to be the case always, and so we learn by experience that

(1) The graph of a function of the first degree is a straight line. For this reason functions of the first degree are often called linear functions.

(2) The graph of a function that is not of the first degree is not a straight line.

As a straight line can be drawn when two points on it are known, we need find only two points when drawing a graph of a function of the first degree. To ensure accuracy the following directions should be noted:

1. Plot three or four points before drawing the straight line. For a slip may have been made in calculating the first two; when the third is plotted it will then be seen that it is not in the same straight line as the others; on plotting the fourth it will probably be found to be in the same straight line as two of the others and the line can be drawn. If no three of the points are on a straight line two or more mistakes have been made, and the previous work should be done over again.

2. The two points actually joined should not be too close together; for if the points are not joined accurately the resulting error is less when the points are far apart.

3. The values of x and y which are plotted should be whole numbers, or should be such that the point representing them is at the intersection of two of the lines on the squared paper. For if not the positions of the points cannot be found exactly and the straight line is certain to be somewhat inaccurate.

* The word curve is often used to include all lines whether straight or bent; it will be used in this general sense throughout the book. The context usually makes it clear whether the word includes straight lines or not.

Example. *Draw the graph of* $y = \frac{3}{5}x - \frac{2}{5}$.

This is of the first degree.

Therefore the graph is a straight line and we need find only two points.

$$y = \frac{3x-2}{5}$$

x	-6	4	9
$3x-2$	-20	10	25
y	-4	2	5

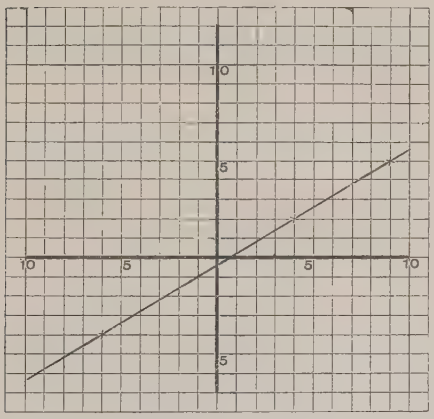


FIG. 21. Scale : 1 side = 1 unit.

Graphs from the Point of View of Geometry.

50. Hitherto we have regarded graphs merely as showing how the value of a function y varies when the value of x changes ; and we have seen that a point can be found which represents any pair of corresponding values of x and y. If we wish to find the point we have had to find the values of x and y.

The subject of graphs is often introduced in the reverse way as being an aid to Geometry.

If the position of a point is known, the point can always be found by taking two measurements.

Geography supplies useful examples. If an explorer finds a mountain that has been previously unknown, he calculates the latitude and longitude of the mountain, with the result that the mountain can afterwards always be found.

In Geometry the measurements taken are (1) the perpendicular distance of the point from a fixed straight line $X'OX$, (2) the distance of the foot N of the perpendicular from a fixed point O on the axis. This point O is shown by drawing a line $Y'OY$ perpendicular to $X'OX$. The length ON, NP are denoted by x and y respectively, the usual rules for signs being observed.

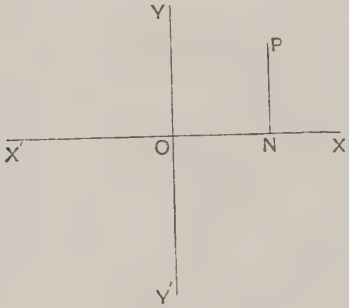


Fig. 22.

x and y are called the co-ordinates of the point P. x, the piece of the axis of x 'cut off', is called the abscissa. y, which shows in what 'row' or 'rank' the point lies, is called the ordinate.

A point whose co-ordinates are $x = 5$, $y = 6$ is referred to as the point (5, 6).

51. If a point moves so as to satisfy some condition it describes a curve which is called in Geometry the 'locus of the point'. As the point moves its co-ordinates change, but, if the point satisfies some condition, its co-ordinates will also satisfy some condition. Conversely, if the co-ordinates are always connected by some relation, the point will always

lie on some curve. In Examples VI d various relations between x and y are given, and the point must lie on the corresponding graph.

Examples VI e.

1. Plot in one figure the following points:
$(4, 5)$, $(-3, 6)$, $(6, -4)$, $(0, 7)$, $(-5, 0)$, $(-6, -5)$.

2. A point moves so that its ordinate is always 2; draw the graph traced out.

3. Plot six different points all having the abscissa equal to 3. Hence find the graph of $x = 3$.

Draw the curve traced out by a point which moves so that its co-ordinates are connected by the relation:

4. $y = x$.
5. $y = 3x - 2$.
6. $y = \frac{4x-1}{3}$.
7. $y = 4x - 1$
8. $y = 4x + 3$.
9. $y = x^2 - 9$.
10. $y = x^3 - x^3$.

CHAPTER VII

SYMBOLIC REPRESENTATION AND SIMPLE PROBLEMS

52. We have already had many examples ot symbolic representation, and the student is by now quite used to the idea that letters may be used as symbols for numbers, and that the operations of addition, subtraction, &c., are represented by the special symbols or signs $+$, $-$, &c.

It must be recollected that letters are used as symbols for numbers, and, if the work deals with concrete quantities, the units must always be mentioned; thus we should not say a man's age is x' but 'a man's age is x years'.

Letters may be used instead of numbers for three different reasons:

(i) To state facts that are true about all numbers, e.g. $a + b = b + a$ states the fact that, whatever two numbers we take

the first added to the second is equal to the second added to the first.

(ii) To stand for some particular numbers whose values are unknown. The letters x, y, z are often used to stand for numbers which are unknown, but whose value is to be found by means of information given in the question. Questions of this kind, which require that a certain unknown number should be found, are called **problems.**

(iii) To state concisely rules for calculating certain concrete quantities. Thus it was shown in § 11 that the area of a rectangle is obtained by multiplying the number of units of length in one side by the number of units of length in the adjacent side. This is stated more concisely thus: $A = lb$ where A stands for the number of units of area, l for the number of units of length in the length, and b for the number of units of length in the breadth.

A concise rule of this kind is called a formula. It is usual in formulae to use letters which suggest what they stand for, e.g. l for number of length units, t for number of time units, &c.

53. In working the following examples the student should read each question carefully, and make sure that he understands it before attempting to answer it. If he cannot see how to start, let him mentally replace the letters by small numbers, so making a simple arithmetic sum. Then he should do with the letters exactly the same as he would with the numbers.

Even though the whole question can be worked in the head, yet sufficient work must be put on paper to show clearly the line of argument.

Example. *A man can buy* x *articles for a certain sum, the price being* a *pence each. How many would he get for the same sum if the price were* $(a+b)$ *pence each?*

[Mental work. A man buys 10 articles at 3*d.* each; how many would he get for the same sum if the price were $(3+2)$ pence each?

Total sum spent $= 10 \times 3 = 30$ pence.

If one cost 5 pence, the number would be $30 \div 5 = 6$.]

Written work.

Total sum spent $= x \times a = ax$ pence.

If one article cost $(a+b)$ pence, he would get for the same sum $ax \div (a+b) = \frac{ax}{a+b}$ articles. *Answer.*

Examples VII a.

1. What is meant by an even number? Prove that the square of any even number is exactly divisible by 4.

2. Find a formula which will show the connexion between the length of the side of a square measured in yards and the area of the square in acres.

3. If a man earns x shillings every week-day, what is his yearly income in pounds?

4. A boy was born in the year A.D. x; how old will he be in the year A.D. y?

5. Prove that when two odd numbers are multiplied together the product is odd.

6. If a train travels u miles in every hour, and if it travels s miles in t hours, find the formula connecting u, s, and t.

7. A man has £x in his pocket, his wife has x shillings, and each of their three children has x pence; how much money have they between them? Express the answer in shillings.

8. During the cricket season a boy scored x runs in y innings and was not out z times. What was his average?

9. How many days will it take a steamer travelling at the rate of v miles per hour to travel s miles?

10. A room is l feet long, b feet broad, and h feet high; find the area of the floor and the area of the walls.

11. How many numbers between 10 and 100 are exactly divisible by 7? Write down the first, second, third, and the rth of those numbers.

12. What is the general algebraic form of a number which can be exactly divided by 5? What do you know about the number $5x+3$?

13. A score of eggs cost x pence. What is the price of y eggs?

14. A cycles at the rate of x miles per hour, B walks at the rate of y miles per hour. How many miles does A gain each hour? If B starts z hours before A, how long will A take to catch B?

15. A workman's wages are a pence per hour for b hours in the week of regular time and c pence per hour for overtime. One week he earns d shillings. How many hours overtime did he work?

16. Prove that the sum of any two numbers multiplied by their difference equals the difference of their squares.

17. What is the value of x sovereigns $2x$ shillings and $(20-3x)$ sixpences? Give the value in sixpences.

18. A garrison have food enough to last 1000 men x days; how long will the food last if the number of men is increased to 3000?

19. If I buy a apples at p pence a dozen and b apples at q pence a dozen, and sell them all at one penny each, how much profit do I make?

20. A room is l feet long, w feet wide, h feet high. What will it cost to whitewash the walls and ceiling at p pence per sq. ft.?

21. A sum of P pounds is invested at simple interest at r % per annum. Find a formula for the interest after t years.

{r % per annum means that the interest on £100 for 1 year is £r.}

22. If x lb. of tea cost y shillings, how many shillings will y lb. of the same tea cost?

23. What should be the price of 7 eggs when a dozen cost x pence?

24. If x ounces of silver are worth y shillings, how many pennyweights of silver are worth z pence?

25. I bought $10x$ yards of cloth at y shillings per yard and sold them at $y-4$ shillings per yard; how many pounds did I lose?

26. If a clock gains x seconds in one day, how many minutes will it gain in y weeks?

27. Prove that the sum of three consecutive numbers is always divisible by 3.

28. If each cubic foot of a substance weighs w pounds, what is the weight of v cubic feet?

29. A body whose volume is v cubic centimetres weighs w grammes, and the average weight of a cubic centimetre is d grammes. What is the formula that connects v, w, and d?

30. If 100 degrees on the Centigrade thermometer are equivalent to 80 degrees on the Réaumur thermometer, how many Centigrade degrees are equal to x Réaumur degrees?

31. Find a formula for all numbers which give remainder 5 when divided by 7. Write down all those numbers between 150 and 200.

54. In problems the question enables us to state that two different expressions are equal to one another; and then by using ordinary common sense the unknown quantity can very often be easily found.

Example I. *A man is three times as old as his brother. In ten years' time he will be twice as old. How old is the brother now?*

Let x years be the brother's age now;
then $3x$ years is the man's age now.
Again $x+10$ years will be the brother's age in 10 years' time;
and $3x+10$ years will be the man's age in 10 years' time.
But the question says that the man's age in 10 years' time will be twice the brother's age.

$\therefore$ $3x+10 = 2(x+10)$.
Remove bracket $3x+10 = 2x+20$.
Take away 10 from each side $3x = 2x+20-10$.
Take away $2x$ from each side $3x-2x = 20-10$,
$x = 10$.

i.e. the brother's present age is 10 years.

Example II. *A sum of two guineas is divided between three men so that the first receives eight shillings more than the second and two shillings less than the third. How much did each receive?*

[Here there are three unknown quantities, but if we find what the first man receives we can easily find what the other men receive.]

Let x shillings be the amount the first man receives;
then $(x-8)$ shillings is the amount the second man receives,
and $(x+2)$ shillings is the amount the third man receives;
then $x+(x-8)+(x+2)$ shillings is the sum distributed.
But the question says two guineas, i.e. 42 shillings is the sum distributed.

Hence $x+(x-8)+(x+2) = 42$.
Remove brackets $x+x-8+x+2 = 42$.
Collect the x terms and the numbers
$3x-6 = 42$.

Add six to each side

$$3x = 42+6, \quad \text{since } -6+6 = 0;$$

i.e. $3x = 48,$

$$x = 16.$$

First man receives 16*s.*, second man 8*s.*, third man 18*s.*

In solving problems

(i) Make sure what the question tells us.

(ii) Make sure what it is required to find.

(iii) Recollect that x stands for a number only, and that the unit must be distinctly stated, e. g. let x feet be the length, let x be the number of men, let x shillings be the money.

Examples VII.

1. Find a number which is 18 less than double the number.

2. Find a number such that half the number is 5 more than one-third of the number.

3. The sum of three consecutive numbers is 57 ; find them.

4. A boy has 6*d.* less than his brother, and 9*d.* more than his sister ; together they have 2*s.* 9*d.* How much has the sister got ?

5. Three stations, A, B, C, are on the same line of railway. The distance from A to B is twice the distance from B to C, and the distance from A to C is 126 miles. How far is it from A to B ?

6. The number of square yards in the area of a certain room is double the number of feet in the length of one of the sides. What is the length of the other side ?

7. Of two numbers, one is three times as great as the other If I take the less number from 16 and add 3 the result exceeds by 7 what I obtain when I subtract the greater number from 30. Find the two numbers.

8. A bill of £12 is paid in sovereigns and half-sovereigns, and 17 coins are used. How many sovereigns are there ?

9. A father's age is 45 years and his son's 9. In how many years will the father be three times as old as the son ?

10. A sum of £100 was contributed by A, B, and C. A contributed £5 more than B, and C £3 less than A. How much did each contribute ?

11. Fourteen years ago a man was twice as old as his son :

at the present time their united ages amount to 100 years. Find the present age of each.

12. The length of a rectangular field is twice its breadth; the length of fence used to enclose the field is 372 yards. Find the area of the field.

13. A post stands with half its length in earth, one-third in water, and 4 feet above the water. How long is the post?

14. A man has in his right-hand pocket three times as many shillings as in his left. If he takes 17 shillings from his right pocket and puts them into the left, he will then have 6 times as many in the left-hand pocket as in the right; how many shillings were there in each pocket at first?

15. Find three consecutive numbers such that three times the greatest shall be equal to twice the sum of the other two.

16. Twenty years hence I shall be seven times as old as I was 28 years ago; how old am I now?

17. Divide £1000 into three parts, so that the first may exceed the second by £350 and the third may be one-half the sum of the other two.

18. A father's age is three times that of his son. Eleven years ago the father was five times as old as his son. What are their ages?

19. The cost of 7 ducks and 11 chickens is 55*s.* and the cost of 13 ducks and 5 chickens is 52*s.* What is the cost of a chicken?

20. A purse contains £9. 10*s.* in sovereigns and half-crowns, and the total number of coins is 20. Find how many sovereigns there are in the purse.

CHAPTER VIII

EQUATIONS

55. In previous examples we have frequently had to find the particular value of x that will make two expressions equal. Questions of this kind occur so often that we must now discuss the usual methods for answering them.

Definitions. An **equation** is a statement that two expressions are equal.

An equation may be true for all values of the letters, e. g. $(x+5)-6 = 2x-(x+1)$ is true for all values of x, since each side is only another form for $x-1$.

Such an equation is called an **identical equation** or **identity**. Very often the sign $\equiv$ is used instead of $=$ when it is known that the equation is an identity.

But usually an equation is true only for some particular value or values of the letters. As such equations are true on certain conditions, they are called **conditional equations**. Unless it is otherwise stated, the word 'equation' may be taken to refer to a conditional equation.

Consider $x-2 = 10$. This equation is certainly not true when $x = 4$, for then $x-2 = 4-2 = 2$; and it will be found that the equation is true only when $x = 12$.

The process of finding for what value of x the equation is true is called **solving** the equation.

A value of x which makes the equation true is called a **root** of the equation, and is said to **satisfy** the equation. The letter whose value is sought is often called the **unknown quantity**.

A **simple, or linear, equation** is one which contains only the first power of the unknown quantity.

56. Axioms.* 1. If equals be added to equals, the sums are equal.

2. If equals be subtracted from equals, the differences are equal.

3. If equals be multiplied by equals, the products are equal.

4. If equals be divided by equals, the quotients are equal.

5. Things which are equal to the same thing are equal to one another.

By continued application of these axioms any equation may be solved.

* Statements which are obviously true, which can be proved only by experience, and which are useful to prove other statements, are called Axioms.

Example. *Solve the equation* $3x-2=8-2x$.

Our object is to reduce this equation to the form $x =$ a number.

$3x-2 = 8-2x.$

Add 2 to each side $3x = 8+2-2x$, since $-2+2 = 0$.

Add $2x$ to each side $3x+2x = 8+2$, since $-2x+2x = 0$.

Collect on each side $5x = 10$.

Divide each side by 5 $x = 2$.

The student should go over the above work very carefully, and find out the reason for adding 2 rather than any other number, for adding $2x$, and for dividing by 5.

57. Verifying the Solution of an Equation. If the work in the last section is correct, when 2 is put instead of x in the equation $3x-2=8-2x$, the two sides should be equal. Verify in the following way:

When $x=2$, $3x-2 = 3.2-2 = 6-2 = 4$.

When $x=2$, $8-2x = 8-2.2 = 8-4 = 4$.

$\therefore$ when $x=2$, $3x-2 = 8-2x$; and the solution is correct.

Method of Work. In solving an equation the following points should be carefully observed:

1. Begin by copying the equation in its original form.

2. Never begin a line with the sign =.

3. As far as possible keep the signs of equality in the same vertical line, one under the other.

4. Always explain on the left hand what you are doing.

Examples VIII a.

By simplifying each side of the following equations, find out whether any of them are identities; if they are not, solve them by using the axioms; in each case verify the solution.

1. $2(x-2) = x-3$.
2. $5-2(x-3) = 3x-9-5(x-4)$.
3. $4+12x = x+15$.
4. $-3+5x = 7x+5$.
5. $3x-(x+7) = x-15$.
6. $80-4(x+1) = 6+x$.
7. $13(2x-1) = 5(5x+7)$.
8. $4(9x-3) = 6+5(6x+3)$.
9. $5x+3-4x+7+2x = -2$.
10. $0 = 3x-5-2(x-2)+1-x$.

Does $x = 5$ satisfy any of the following equations?

11. $2x-3(4-x) = 7+x$.

12. $(x-2)(x-3) = (x-4)(x+1)$.

13. $x^2-4x-2 = 3+5x-x^2$.

14. What value of x will make $x-7 = 0$?

15. What value of x will make $x-6$ twice as much as $4-3x$?

16. What is the value of a if $x = 5$ is a solution of

$$x^2-4ax+15 = 0?$$

17. If $x = -2$, what must be the value of y in order that $3x-4y = 5$?

18. If $y = -1$ and $z = +2$, what must be the value of x in order that $2x+4y-7z = 24$?

19. Is the equation $(y-z)^2+(z-x)^2 = (x-y)^2$ satisfied by

$$x = 4,\ y = 3,\ z = 2?$$

20. What value of x will make $x-a = a-x$?

58. Transposing. Let us look again at the work in § 56.

The original form of the equation is $3x-2 = 8-2x$.

After two steps the equation becomes $3x+2x = 8+2$.

Comparing these two forms it is seen that the second form can be got from the first by taking the 2 across from the left-hand side to the right-hand side, and at the same time changing its sign, and also taking the $2x$ across from the right-hand side to the left-hand side, and at the same time changing its sign. Very likely the student has already noticed similar results when working the last set of examples. Taking a quantity across from one side to the other is called **transposing.**

Rule for Transposing. Any quantity may be transposed from one side to the other if its sign is changed at the same time.

59. Fractions. If there are fractions on either side of an equation they can be removed by applying axiom 3, § 56.

Example. *Solve* $\frac{x}{8}-\frac{x-2}{5}+\frac{1}{10}(x-22)+\frac{x-12}{20} = \frac{32-x}{40}$.

$$\frac{x}{8} - \frac{x-2}{5} + \tfrac{1}{10}(x-22) + \frac{x-12}{20} = \frac{32-x}{40}.$$

Multiply by 40 the L. C. M. of the denominators

$$\frac{x}{8}\times 40 - \frac{x-2}{5}\times 40 + 40\times \tfrac{1}{10}(x-22) + 40\times\frac{x-12}{20} = \frac{32-x}{40}\times 40.$$

Cancel $5x-8(x-2)+4(x-22)+2(x-12) = 32-x.$

Remove brackets $5x-8x+16+4x-88+2x-24 = 32-x.$

Collect on each side $11x-8x+16-112 = 32-x.$

i.e. $3x-96 = 32-x.$

Transpose x terms to the left, numbers to the right.

$$3x+x = 32+96.$$

Collect $4x = 128.$

Divide by 4 $x = 32.$

Verification.

If $x = 32$, left-hand side $= \frac{32}{8} - \frac{30}{5} + \frac{10}{10} + \frac{20}{20}$

$= 4-6+1+1$

$= 0;$

and right-hand side $= \frac{0}{40}$

$= 0.$

$\therefore x = 32$ is the correct solution.

Literal Equations. It sometimes happens that an equation contains not only the letter whose value is to be found, but also other letters. The values of these letters are supposed to be known, and the letters are transposed to the right-hand side like ordinary numbers.

Example. *Solve* $(x-a)+(x-b)-(c-x) = 0.$

Remove brackets $x-a+x-b-c+x = 0.$

Transpose and collect $3x = a+b+c.$

Divide by 3 $x = \frac{a+b+c}{3}.$

Example. *If* $3y+4x = 7$, *find* y *in terms of* x.

Here y is the unknown quantity and x is treated as a known quantity.

$$3y+4x = 7.$$

Transpose $3y = 7-4x.$

Divide by 3 $y = \frac{7-4x}{3}.$

Examples VIII.

Solve the following equations and verify the solutions:

1. $7x-5x+3x=15$.
2. $5+9x=2x+19$.
3. $32x-7=4x+5+44$.
4. $-2x-5-6x=8x+21-6-6x$.
5. $5x-7+9-4x-3=7-2x+3x-8$.
6. $2x-(x-7)=15$.
7. $3(x+2)-3(x-2)=x+12$.
8. $4(x-3)+5(x-4)+6(x-1)=7$.
9. $4(2-x)-3(2x-3)=7(2+x)-16x-1$.
10. $2(3+2x)+4(6+5x)+3(x+1)+4(x+1)=6$.
11. $(x-4)(x-5)=(x-6)(x-2)$.
12. $(x+5)(x-4)=(x-3)(x+2)-(x-2)-1$.
13. $3\{x-2(x-4)\}=2(x-5)-1$.
14. $2(x-3)(x-4)=2(x-6)(x-1)-(2x-4)$.
15. $3-\{x-2(3x-4)\}=x^2-(x-1)(x+1)+4$
16. $\frac{x}{12}-\frac{x}{16}=\frac{x}{6}+\frac{7}{12}$.
17. $\frac{1}{2}x-\frac{1}{3}x+7=\frac{5}{6}x-5$.
18. $\frac{x}{3}-\frac{x}{4}+\frac{1}{6}=\frac{x}{8}+\frac{1}{12}$.
19. $\frac{3x+1}{3}=\frac{x-1}{4}$.
20. $\frac{x-1}{2}+\frac{x-3}{3}=1$.
21. $5(x-2)-3(3x-4)=17-7x$.
22. $2(x-3)-3(x+1)+2(x-1)+4=0$.
23. $(x+2)(x+3)=(x+4)(x+5)+2$.
24. $(x-1)(x-3)+(x-2)^2=2(x-2)(x-3)$.
25. $\frac{4x-5}{9}-\frac{3x-5}{11}=\frac{2x-5}{4}$.
26. $x(a+2)-6=a(x+6)-2$.
27. What number is that which exceeds its sixth part by 10?
28. If ducks' eggs cost 4*d.* a dozen more than hens' eggs, determine the price of each per dozen, when 7 ducks' eggs and 19 hens' eggs can be bought for two shillings.
29. A man sold a horse for £35 and half as much as it cost him, and by doing so he gained £10. What was the original price of the horse?

30. Find the number the half of which exceeds the difference of its third and fourth parts by ten.

31. A bag contains 129 coins, some of which are sovereigns and the rest shillings; they amount altogether in value to £41. 12*s*. How many coins of each kind are there?

32. A man has been saving annually a quarter of his income. His income is increased by £100, and he now saves a third of his income and finds that he is saving annually £40 more than before. What was his original income?

33. A man performed a journey of 7 miles in $1\frac{1}{4}$ hours. He walked part of the way at the rate of 4 miles an hour, and he rode the rest of the way at the rate of 10 miles an hour. How many miles did he walk?

Solve the following equations, and in each case verify the solution:

34. $\frac{x+1}{2} - \frac{x+3}{5} + 2\frac{1}{2} = 0.$

35. $\frac{1}{2}(9-2x) = 1\frac{1}{2} - \frac{1}{10}(7x-18).$

36. $\frac{2x+4}{3} - 2(x-1) = 7\frac{1}{3} - 2x.$

37. $\frac{3}{2}x - \frac{1}{3}(4-x) = 2\frac{1}{3} - 3(x-2).$

38. $\frac{2x+1}{3} - \frac{x+6}{5} = x-3.$

39. $\frac{3x+1}{7} - \frac{x+8}{5} + \frac{x+1}{3} = 0.$

40. $\frac{-4x+5}{9} + \frac{6x+1}{8} + \frac{1}{12} = 0.$

41. $\frac{2x-1}{3} - \frac{3x-1}{2} + \frac{5x-1}{6} = 0.$

42. $\frac{7x-4}{15} - 2 + \frac{3x+3}{16} = \frac{7x+1}{20}.$

43. $\frac{x}{3} - 8 + \frac{x+40}{5} = 47 - \frac{x}{4}.$

44. $\frac{x-a}{3} - a = \frac{x-b}{5} - 3b.$

45. A poulterer buys a eggs at the rate of b shillings a hundred. One-third of them are broken before he receives them. At what rate per dozen must he sell them so as neither to gain nor lose?

6. A sum of $2a$ pence is to be divided between b boys and c girls, so that for every penny each boy receives each girl is to receive three halfpence. What will be the share of each boy?

47. A farmer bought a certain number of sheep for forty guineas. He lost one-tenth of them, and sold the rest for half a guinea a head more than he had given for them, and thus gained five guineas. Find the number of sheep he bought.

48. If $3x+2y=3\frac{1}{2}$, find the value of y in terms of x, and hence find the value of y when $x=-2\frac{1}{2}$.

49. Find x in terms of y when

$$3x-4(x-2y)=\frac{3x-4y}{3}-2.$$

50. What is the value of a if $x=-3$ satisfies

$$\frac{2-x}{3}+\frac{a+2x}{4}=\frac{x-3}{5}?$$

51. Find the value of A in order that

$$3(x-3)+4(2x-5)-3(x-2)=A(x-1)-15$$

may be an identity.

52. If $\frac{2(x-1)}{5}-\frac{11(x-2)}{\quad}=1$ is satisfied by $x=\frac{1}{5}$, find the missing denominator.

53. Solve $\frac{1}{3x}-\frac{1}{4x}=\frac{1}{24}$.

54. Solve $(x+2\frac{1}{2})(x-1\frac{1}{2})-(x-3)(x-5)=-\frac{5}{4}$.

55. Given $2y=5x-6$, find the value of y when $x=3{\cdot}2$ and when $x=-2{\cdot}4$.

56. Solve $2x-5\{3x-7(4x-9)\}=66-9(x-3)$.

57. Solve $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}=\frac{1}{6}$.

58. A cruiser running at the rate of 16 miles an hour is pursuing an enemy's ship 18 miles off running at the rate of 12 miles an hour. How many miles can the latter run before she is overtaken?

59. A has a dozen more apples than B. If A were to give one-fourth of his apples to B, B would have 6 more than A. How many has A?

60. A train travelling 40 miles an hour takes two hours less in going a certain distance than a train travelling 24 miles an hour. What is the distance?

CHAPTER IX

SIMULTANEOUS EQUATIONS. GRAPHICAL SOLUTIONS

60. If an equation contains two unknown quantities it may be satisfied by many pairs of values. The equation $x+y=50$ is satisfied by $x=1,\ y=49$; $x=3,\ y=47$; $x=-3,\ y=53$; and many other values. Similarly the equation $x-y=7$ has a large number of solutions, and it may be necessary to find what values of x and y will satisfy both equations.

When two or more equations have to be satisfied at the same time, that is, when they are to be satisfied by the same values of the unknown letters, they are said to be **simultaneous.**

Example. *The sum of the lengths of two posts is* 50 *feet, and one is* 3 *feet longer than the other. Find their lengths.*

Let x feet be the length of the longer and y feet be the length of the shorter; then from the question

$$x+y=50,$$
$$x-y=3.$$

Since these equations are to be satisfied by the same values of x and y they are simultaneous.

61. In all methods of solving simultaneous equations the object in view is to eliminate or get rid of one of the unknown letters; the resulting equation with only one unknown is then solved by the methods explained in the last chapter.

62. Method I.

Obtain two equations in each of which one of the unknowns has the same numerical coefficient.

Example I. *Solve* $2x+3y=8,$ (i)

$$5x-4y=-3.$$ (ii)

Multiply equation i by 4. $8x+12y=32.$ (iii)

Multiply equation ii by 3. $15x-12y=-9.$ (iv)

Add the last two equations

$$23x = 23.$$

$$\therefore \quad x = 1.$$

Substitute 1 instead of x in equation i

$$2 + 3y = 8.$$

Transpose $3y = 6,$

$$y = 2.$$

Answer. $x = 1,\ y = 2.$

Verification. Equation i. $2 \times 1 + 3 \times 2 = 2 + 6$

$$= 8.$$

Equation ii. $5 \times 1 - 4 \times 2 = 5 - 8$

$$= -3.$$

Example II. *Solve* $21x - 19y = 101,$ (i)

$$35x + 23y = 59.$$ (ii)

Multiply equation i by 5. $105x - 95y = 505.$

Multiply equation ii by 3. $105x + 69y = 177.$

Subtract $-164y = 328.$

$$\therefore \quad y = -2.$$

Substitute in equation i $21x + 38 = 101.$

Transpose $21x = 63,$

$$x = 3.$$

Answer. $x = 3,\ y = -2.$

Examples IX a.

Solve (and in each case verify) the following simultaneous equations:

1. $3x + 5y = 11,$
   $6x - 5y = 7.$

2. $6x + 17y = 28,$
   $5x + 13y = 21.$

3. $-2x + 7y = 27,$
   $13x - 4y = 32.$

4. $15x + 21y = 12,$
   $21x - 35y = -112.$

5. $x - 3y = -2,$
   $4x - 9y = 7.$

6. $5x + 3y = 120,$
   $10x = 9y + 90.$

7. $9x + 24y = 26,$
   $17x - 3y = 33.$

8. $5x - 7y = 22a,$
   $4x + 23y = -11a.$

9. $ax - by = 3ab,$
   $ax + by = 5ab.$

10. $\frac{x}{4} + \frac{y}{3} = 8,$
    $\frac{x}{3} - \frac{y}{5} = 1.$

11. $2x + y + 12 = 0,$
    $x - 6y + 19 = 0.$

12. $x + 3y = 4x + y = 2\frac{1}{2}$

63. *Solve* $7x-6y=33,$ (i)

$3x+11y=-13.$ (ii)

Method II.

From equation i $7x=33+6y.$

$$\therefore \quad x=\frac{33+6y}{7}. \tag{iii}$$

From equation ii $3x=-13-11y.$

$$\therefore \quad x=-\frac{13+11y}{3}.$$

But since x and y have the same values in both equations these values of x must be the same.

Hence $$\frac{33+6y}{7}=-\frac{13+11y}{3}.$$

Multiply by 21 $99+18y=-91-77y.$

Transpose $95y=-190,$

$y=-2.$

Substitute in iii $$x=\frac{33-12}{7}$$

$=3.$

Answer. $x=3,\ y=-2.$

Method III.

From equation i $7x=33+6y.$

$$\therefore \quad x=\frac{33+6y}{7}. \tag{iii}$$

Substitute for x in equation ii

$$\frac{3\,(33+6y)}{7}+11y=-13.$$

Multiply by 7 $99+18y+77y=-91.$

Transpose $95y=-190.$

$\therefore \quad y=-2.$

Substitute in iii $x=\dfrac{33-12}{7}=3.$

Examples IX b.

Use Methods II and III to solve the following simultaneous equations, and in each case verify:

1. $x+7y=26,$

$\dfrac{x}{5}+\dfrac{y}{3}=2.$

2. $7x-5y+6=13y-5x-6,$

$2y-x=2.$

3. $\frac{x}{3}+\frac{y}{7}=1$,
$10x-4y=1$.

4. $\frac{x}{5}+\frac{y}{8}=-1$,
$7x+2y=3$.

5. $3x=4y-7$,
$3y=4x-7$.

6. $x=\frac{5y-7}{3}+2$,
$y=\frac{3x-1}{4}-2$.

7. $x=a(y-b)+b=b(y-b)+a$.

8. $\frac{x+1}{2}=8-\frac{y-1}{3}$.
$\frac{y+1}{2}=9-\frac{x-1}{3}$.

9. $\frac{x}{2}+y=9$, $\frac{x}{6}=y-1$.

10. $x-3=\frac{y-5}{6}$,
$3y-2=4(x-5)$.

Graphical Method of solving Simultaneous Equations.

64. It has been shown in Chapter VI that if x and y satisfy an equation, the point whose co-ordinates are x and y will lie on a definite curve. This curve is called the **graph** of the equation.

If a point lies on two curves, that is, if it is the point where two curves meet, its co-ordinates will satisfy two equations. And, if x and y satisfy two equations, the corresponding point will be at the intersection of the graphs of the equations.

Example. *Solve* $2x+3y=8$, (i)
$5x-4y=-3$. (ii)

Both equations are of the first degree, therefore the graphs will be straight lines (see § 49).

From equation i $y=\frac{8-2x}{3}$;

if $x=$	1,	−5	7,
$y=$	2.	6	−2.

From equation ii $y=\frac{5x+3}{4}$;

if $x=$	5,	−7	9,
$y=$	7,	−8	12.

These graphs are shown in Fig. 23. It is seen that they intersect at the point whose co-ordinates are $x = 1$, $y = 2$. Hence these values must satisfy both equations.

Answer. $x = 1$, $y = 2$.

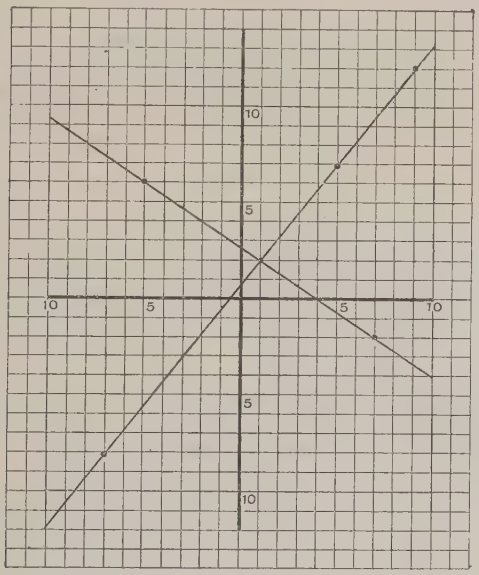


Fig. 23. Scale : 1 side = 1 unit.

65. The graphical method is especially useful when it is not required to have an absolutely correct answer; for practical purposes an answer correct to two significant figures is often quite near enough.

Example. *Solve* $8x + 11y = 17$, (i)

$13x - 19y = 23$. (ii)

The equations are of the first degree, therefore both graphs are straight lines.

From equation i $\quad x = \dfrac{17 - 11y}{8}$;

if $y =$	-5,	11 19,
$x =$	9,	-13 -24.

From equation ii $x = \dfrac{23+19y}{13}$;

if $y = 0$,	10	-10,
then $23y+19 = 23$	213	-167,
$x = 1{\cdot}77$	16·38	$-12{\cdot}85$.

It is better to find whole number values: but with awkward numbers this might take a good deal of time. The answer is wanted only approximately, so that we may be content with approximate values for the plotted points.

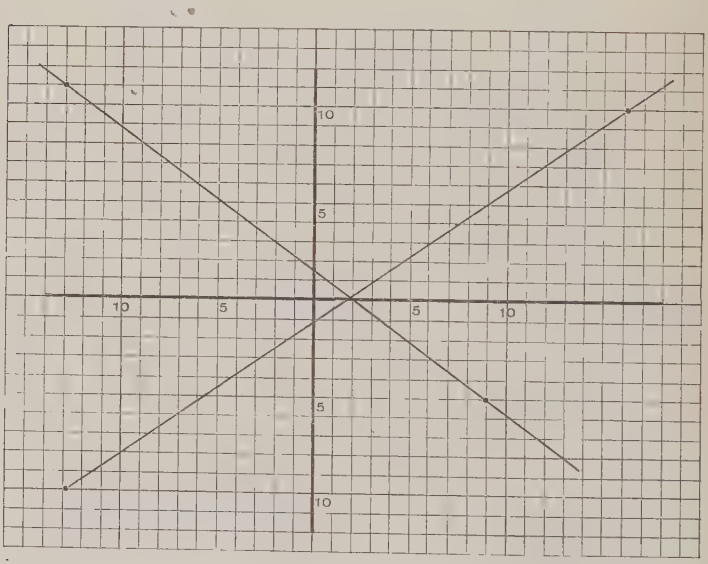


Fig. 24. Scale: 1 side = 1 unit.

It is seen from Fig. 24 that the graphs intersect at a point where $x = 2$ almost exactly and $y =$ rather more than ·1.

Answer. $x = 2{\cdot}0$ $y = {\cdot}1$.

The correct answers are $x = 2 - \frac{14}{295}$, $y = \frac{37}{295}$

$= 2 - {\cdot}04$, $y = {\cdot}12$.

Some care is necessary in estimating the decimal place. The side of the square can by eye be divided into four equal

parts. If the length is just over the half it is reckoned as ·6, if just less as ·4; if just over the first quarter ·3. if just less ·2, &c.

66. It should be noted that the graphical method may be applied when the graphs are not straight lines, the solution always being given by the co-ordinates of the point of intersection. If there is more than one point of intersection, there is more than one solution; if there are no points of intersection, there are no solutions.

Examples IX c.

Solve by graphs (and verify by solving by one of the other methods or by substitution):

1. $3x-5y=-34$, $2x+8y=34$.
2. $7x-3y=27$, $4x+5y=2$.
3. $2x+y=8$, $x+y=1$.
4. $9x-2y=-34$, $3x+11y=82$.
5. $2-3x=19y$, $4x-9y=87$.

In the following examples find the answers correct to the first decimal place:

6. $3x-4y=5$, $-5x+6y=3$.
7. $2x+11y=25$, $7x-4y=12$.
8. $3x-y=9$, $13x+17y=2$.
9. $2x+6y=15$, $3x-5y=13$.
10. $13x-25y=14$, $11x+19y=18$.
11. $3x-4y=10$, $x^2+y=1$.
12. $y=x^2-2x$, $2y=1$.

67. Further hints on solving simultaneous equations.

1. Either or both of the equations may have to be simplified by clearing of fractions, removing brackets, &c.

2. If $A=B=C$, then $A=B$, $A=C$, $B=C$.

Simultaneous equations are sometimes set in this form.

Example. *Solve* $x-7y-2=3x-4y+6=3-2x$.

The first step is to write down two separate equations, e.g.

$$x-7y-2=3-2x, \quad \text{(i)}$$
$$3x-4y+6=3-2x. \quad \text{(ii)}$$

Simplify each of them:

$$3x-7y=5, \quad \text{(iii)}$$
$$5x-4y=-3. \quad \text{(iv)}$$

Finish by either of Methods I, II, III.

3. The equations may contain $\frac{1}{x}$ and $\frac{1}{y}$ instead of x and y.

Example. *Solve* $\frac{3}{x}+\frac{4}{y}=18,$ (i)

$$\frac{5}{x}-\frac{3}{y}=1. \quad \text{(ii)}$$

Treat the numerators exactly as the coefficients are treated in Method I.

Equation i × 3	$\frac{9}{x}+\frac{12}{y}=54.$
Equation ii × 4	$\frac{20}{x}-\frac{12}{y}=4.$
Add	$\frac{29}{x}=58.$
Multiply by x	$29=58x.$
	$\therefore \quad x=\frac{29}{58}=\frac{1}{2}.$
Substitute in i	$6+\frac{4}{y}=18$
	$\frac{4}{y}=12.$
	$\therefore \quad y=\frac{1}{3}.$

4. If the coefficients are letters or contain letters, the methods adopted are exactly the same as with numerical coefficients.

Example. *Solve* $px+qy=r,$ (i)

$$qx-py=0. \quad \text{(ii)}$$

Equation i × p	$p^2x+pqy=pr.$
Equation ii × q	$q^2x-pqy=0.$
Add	$(p^2+q^2)x=pr.$
	$\therefore \quad x=\frac{pr}{p^2+q^2}.$

We could find y by substituting for x in the simpler of the original equations, in this case equation ii. But with letter coefficients it is often easier to begin again and eliminate x.

Equation i $\times q$	$pqx+$	$q^2y = qr.$
Equation ii $\times p$	$pqx-$	$p^2y = o.$
Subtract	$(p^2+q^2)\,y$	$= qr$

$$y = \frac{qr}{p^2+q^2}.$$

Answer. $x = \dfrac{pr}{p^2+q^2}$, $y = \dfrac{qr}{p^2+q^2}$.

68. We may have more than two unknowns. In order to obtain definite solutions we must have as many equations as unknowns.

Example. *Solve* $2x-3y+4z=20,$ (i)

$3x+4y-5z=-20,$ (ii)

$4x-5y-6z=-4.$ (iii)

Eliminate one of the letters from two of the equations:

i $\times 5$	$10x-15y+20z =$	$100.$
ii $\times 4$	$12x+16y-20z =$	$-80.$
Add	$22x+\ y\ =$	$20.$ (iv)

Eliminate the same letter from another pair of the equations:

i $\times 3$	$6x-\ 9y+12z =$	$60.$
iii $\times 2$	$8x-10y-12z =$	$-8.$
Add	$14x-19y\ =$	$52.$ (v)

iv and v are a pair of simultaneous equations and can be solved by one of the methods already shown.

From iv	$y =$	$20-22x.$ (vi)
Substitute in v	$14x-380+418x =$	$52.$
Transpose and collect	$432x =$	$432.$
	$\therefore\ x =$	$1.$
Substitute in vi	$y =$	$20-22 = -2.$

Substitute in any of the original equations.

Substitute in i	$2+6+4z =$	$20.$
Transpose and collect	$4z =$	$12,$
	$z =$	$3.$

Answer. $x = 1$, $y = -2$, $z = 3$.

Examples IX d.

Solve:

1. $x-\frac{2}{5}y=3\frac{1}{2}$,
   $\frac{4}{5}x+y=-4\frac{1}{3}$.

2. $3(x-2)-2(y+3)=1$,
   $2(x-3)+3(y+2)=0$.

3. $\frac{3}{x}-\frac{4}{y}=2$,
   $\frac{4}{x}+\frac{5}{y}=13$.

4. $\frac{4}{x}-\frac{5}{y}=\frac{2}{a}$,
   $\frac{3}{x}+\frac{6}{y}=\frac{21}{a}$.

5. $ax+by=a^2-b^2$,
   $2bx-3ay=5ab$.

6. $17x+19y=20$,
   $19x+17y=16$.

7. $4x-3y-z+1=0$,
   $2x+2y-z-18=0$,
   $x+2y+z-11=0$.

8. $\frac{6x-2y}{11}-1=\frac{2x-y}{5}$.
   $\frac{2x+3y}{3}-3=\frac{5x-2y}{2}$

9. $\frac{3}{x}-\frac{4}{y}=\frac{5}{6}$,
   $\frac{5}{x}+\frac{6}{y}=3\frac{1}{2}$.

10. $5x-\frac{5y+2}{4}=32$,
    $3y-\frac{x+2}{3}=9$.

11. $ax-by+c=0$,
    $b(x-y)=a-b$.

12. $px+qy=qx+py=r$.

13. $3x-4y+7z=7$,
    $2x-5y-6z=7$,
    $8x+3y-10z=5$.

14. $2x-4y=2$,
    $3y-5z=13$,
    $x+y+z=2$.

15. $x+y-z=a$,
    $x-y+z=b$,
    $-x+y+z=c$.

16. $ax+by=ab$, $\frac{x}{a}+\frac{y}{b}=\frac{1}{ab}$.

17. $x+y=1$, $ax+by=a+b$.

18. $\frac{9}{x}-\frac{4}{y}=8$, $\frac{13}{x}+\frac{7}{y}=101$

19. $\frac{x-a}{2}+\frac{y-b}{3}=a$, $\frac{x-b}{3}+\frac{y-a}{2}=b$.

20. $\frac{x-y}{3}+\frac{x+y}{5}=4$, $\frac{x-y}{5}-\frac{x+y}{9}=1\frac{1}{15}$.

21. $x+2y+z+7=0$,
$2x+y=z+1$,
$3x+2z=y+2$.

22. $\frac{2x-y}{3}=\frac{x+y+1}{4}=\frac{2y-x}{5}$.

23. $8x+7y=2$, $392x-336y=1$.

24. $4x+5y+z=6$,
$x+7y+2z=10$,
$5x-3y-6z=16$.

25. $3x-2y-z=8$,
$x+3y-2z=4$,
$7x-y-4z=20$.

26. $2x+y-3z=9$,
$3x-2y+z=3$,
$13x-4y-3z=19$.

Examples IX.

1. The sum of two numbers is 176 and their difference 48; find the numbers.

2. Three times one number exceeds another by 17; four times the second number exceeds twice the first by two. What are the numbers?

3. If $3x+4y=7=y-x$, find the value of $4x+3y$.

4. Solve $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}=1+\frac{y}{c}$.

5. Solve $\frac{3(x+1)}{7}+\frac{4}{y}=-5$,
$\frac{4(x-1)}{5}-\frac{5}{y}=14$.

6. A sum of £5 is divided between two men, so that one receives eight times as many pence as the other receives shillings. What sum does each receive?

7. *A* has twice as much money as *B* to start with, and saves three times as much as *B* every week. At the end of 20 weeks *A* has £13 and *B* has £6. How much does each save per week?

8. If 5 lb. of tea and 7 lb. of coffee cost £1. 3*s*. 4*d*., and 8 lb. of tea and 3 lb. of coffee cost £1. 3*s*. 8*d*., find the cost of tea and coffee per lb.

9. A page of print contains 1300 words if in large type and 1850 if in small type. If an article of 18625 words occupies exactly 12 pages, how many pages are in small print?

10. The graph of the equation $ax+by=1$ passes through the two points $(3, -2)$ and $(-1, 2)$. Find the values of a and b.

11. What must be the values of m and c if the graph of $y = mx + c$ passes through the two points $(4, -2)$ and $(-3, 2)$?

12. Find the equation of a straight line which cuts the axis of x where $x = 3$ and also passes through the point $(4, 5)$.

13. Find the values of a, b, and c in order that the graph of $y = ax^2 + bx + c$ may pass through three points $(-1, 2)$, $(3, 4)$, $(2, -2)$.

14. The age of a father is twice the sum of the ages of his two sons, the elder of whom is twice as old as the younger. Nine years hence the father will be three times as old as his younger son. Find their ages.

15. A man buys two horses for £90. By selling one for four-fifths of its cost price and the other for five-fourths of its cost price he makes a profit of £6 on the whole transaction. Find the cost price of each horse.

16. If 3 be taken both from the numerator and from the denominator of a certain fraction, the resulting fraction is equal to $\frac{2}{3}$, but if 3 be taken from the numerator and added to the denominator of the same fraction, the resulting fraction is equal to $\frac{1}{3}$. Find the original fraction.

CHAPTER X

MULTIPLICATION

69. It has already been shown that $xy = yx$; it follows that, when there are more than two factors, the multiplications may be performed in any order, e. g.

$$abcd = badc = bdac.$$

70. Suppose it is required to multiply together two different powers of the same quantity, e. g. to find the product of x^3 and x^5.

$$x^3 = 1 \times x \times x \times x,$$
$$x^5 = 1 \times x \times x \times x \times x \times x.$$
$$\therefore\ x^3 \times x^5 = 1 \times x \times x \times x \times 1 \times x \times x \times x \times x \times x$$
$$= 1 \times 1 \times x \times x \times x \times x \times x \times x \times x \times x \quad \text{by § 69}$$
$$= 1 \times x \times x \times x \times x \times x \times x \times x \times x$$
$$= x^8.$$

Other cases could be done in the same way, and it is seen that **when two powers of x are multiplied together, the product is a power of x whose index is the sum of the original indices.**

The same rule applies for a number or for any letter or for any algebraical quantity.

$5^3 \times 5^4 = 5^7$, $a^2 \times a^3 = a^5$, $(x+a)^2 \times (x+a)^4 = (x+a)^6$.

71. Each of the two powers may be preceded by a coefficient.

Example I. *To find the product of* $3x^3$ *and* $7x^5$.

$$\begin{aligned} 3x^3 \times 7x^5 &= 3 \times x^3 \times 7 \times x^5 \\ &= 3 \times 7 \times x^3 \times x^5 \quad \text{by § 69} \\ &= 21x^8 \quad \text{by § 70.} \end{aligned}$$

The same reasoning can be applied to all cases.

Example II. *To find the product*

$$-2a^2bc^3 \times 3ab^2c^2 \times 4a^3bc^4.$$

$$\begin{aligned} \text{Product} &= -2a^2bc^3 \times 3ab^2c^2 \times 4a^3bc^4 \\ &= -2 \times 3 \times 4 \times a^2 \times a \times a^3 \times b \times b^2 \times b \times c^3 \times c^2 \times c^4 \\ &= -24a^6b^4c^9. \end{aligned}$$

Examples X a.

Find the following products:

1. $2x \times 3x^3$.	2. $2a \times 3x^3$.	3. $2abc^3 \times 3ab^3c^4$.
4. $5x^2 \times 3x^4$.	5. $7a^2 \times 4a^3$.	6. $4ax^2 \times 6a^2xy$.
7. $2y^3 \times 7y^5$.	8. $9z^2 \times 3xyz$.	9. $3xy^2 \times 5x^2y$.
10. $3xy \times 2yz \times xz$.	11. $3xy^2 \times 5x^2y^4$.	12. $-2ab \times abc^2$.
13. $4x^{15} \times 15x^4$.	14. $3p^2q \times 7pq^3$.	15. $3a \times 4b \times 5c$.
16. $7x^3y \times 8xy^4$.	17. $x^5 \times x^{19} \times x^{15}$.	18. $3a^2b^5 \times 4a^3b^4 \times (-b^2)$.
19. $8a^2 \times 9b^2 \times 6c^3$.	20. $3abc \times 4xyz$.	21. $21a^2 \times 2b^{21} \times 3a^2b$.

What must $2a^2x^3$ be multiplied by to give the following products?

22. $6a^3x^5$. 23. $-8a^4x^7$. 24. $4a^4x^9$.

72. The simpler cases of Multiplication have already been dealt with in the chapter on Brackets.

Example. *Multiply* $3x^3-2x^2+x-4$ *by* $2x$.

$$\begin{aligned}\text{Product} &= 2x(3x^3-2x^2+x-4)\\ &= 6x^4-4x^3+2x^2-8x,\end{aligned}$$

since the quantity outside the bracket multiplies every term inside. The work can also be set down as follows very much in the same way as in Arithmetic.

$3x^3-2x^2+x-4$		4 3 5 4	
$2x$		2	
$6x^4-4x^3+2x^2-8x$	Product.	8 7 0 8	

It is usual to place the multiplier immediately under the left-hand term in Algebra, although it does not really matter where it is placed. There is a good deal to be said for arranging the work so that all like powers come in the same column, as follows:—

$$\begin{array}{r} 3x^3-2x^2+\ x-4 \\ 2x \\ \hline 6x^4-4x^3+2x^2-8x \end{array}$$ Product.

73. Mental Multiplication. It is shown in § 39 that $(x+a)(x+b)=x^2+(a+b)x+ab$; it should be noticed that the coefficient of x in the product is the algebraic sum of the two constants in the factors (either a or b, or both, may be negative) and the constant in the product is the product of the original constants. Hence all similar multiplications can be done mentally.

$$\begin{aligned}(x+3)(x+5) &= x^2+(5+3)x+5\times 3\\ &= x^2+8x+15,\\ (x-3)(x+5) &= x^2+(-3+5)x+(-3)\times 5\\ &= x^2+2x-15.\end{aligned}$$

The process is easily reversed.

Example. *What two expressions multiplied together give a product:*

(i) $x^2-8x+12$, (ii) $x^2-8x-20$?

(i) $x^2-8x+12 = x^2-(6+2)x+(6\times 2)$
$= (x-6)(x-2).$

(ii) $x^2-8x-20 = x^2+(-10+2)x+(-10)\times(+2)$
$= (x-10)(x+2).$

Examples X b.

Find the product of:

1. x^2-2x-1 and $3x$.
2. $3y^3-2y^2+4y-1$ and $2y^2$.
3. $3a^3-3a^2-2$ and $4a^2$.
4. $z^4-2z^3+4z^2-3z+2$ and $4z^3$.
5. $5a^4-4a^3+a-2$ and $-3a^3$.
6. $x-x^3+2x^2-4$ and $3a$.
7. $9x^4-3x^2+2-x$ and $4x^4$.
8. $k^4-k^3+k^2-k+1$ by $-k^2$.

Find mentally the products of:

9. $x+5$ and $x+3$.
10. $x-5$ and $x+3$.
11. $x-2$ and $x+7$.
12. $y-3$ and $y-4$.
13. $x-4$ and $x+6$.
14. $x-6$ and $x-8$.
15. $x+5$ and $x-5$.
16. $x+5$ and $x-9$.
17. $x+7$ and $x-7$.
18. $y+3$ and $y-3$.
19. $a+2$ and $a+2$.
20. $a+2$ and $a-2$.
21. $x+12$ and $x-10$.
22. $x-14$ and $x+7$.
23. $x-6$ and $x+6$.
24. $2x-5$ and $2x+5$.
25. $y-8$ and $y-2$.
26. $a-15$ and $a-11$.
27. $z-3$ and $z-7$.
28. $x-1$ and $x+50$.
29. $x-5$ and $x+8$.
30. $x+6$ and $x+9$.

What quantities multiplied together give the following products?

31. $x^2+8x+12$.
32. $x^2+7x+12$.
33. $x^2-8x+12$.
34. $x^2-7x+12$.
35. $x^2-10x+9$.
36. $x^2+7x+10$.
37. x^2-3x-4.
38. $x^2-5x-14$.
39. x^2+3x-4.
40. $x^2+5x-14$.
41. x^2-4.
42. x^2-9.

74. Example. *Multiply* $3x^3-2x^2+x-4$ *by* $2x-3$.

Product $= (2x-3)(3x^3-2x^2+x-4)$
$= 2x(3x^3-2x^2+x-4) - 3(3x^3-2x^2+x-4)$
$= 6x^4-4x^3+2x^2-8x$
$\quad -9x^3+6x^2-3x+12$
$= 6x^4-13x^3+8x^2-11x+12.$

Either of the following arrangements may be used:

$$\begin{array}{l} 3x^3 - 2x^2 + x - 1 \\ 2x - 3 \\ \hline 6x^4 - 4x^3 + 2x^2 - 8x \\ \quad - 9x^3 + 6x^2 - 3x + 12 \\ \hline 6x^4 - 13x^3 + 8x^2 - 11x + 12 \quad \text{Product.} \end{array}$$

$$\begin{array}{r} 3x^3 - 2x^2 + x - 4 \\ 2x - 3 \\ \hline 6x^4 - 4x^3 + 2x^2 - 4x \\ \quad - 9x^3 + 6x^2 - 3x + 12 \\ \hline 6x^4 - 13x^3 + 8x^2 - 7x + 12 \quad \text{Product.} \end{array}$$

Example. *Find the product of* $3 - x^3 + 4x^2 - 3x$ *and* $2x^3 + x - 2x^2 + 3$.

Before multiplying, both expressions must be arranged in descending powers of x, or in ascending powers.

(i) Descending powers.

$$\begin{array}{l} 2x^3 - 2x^2 + x + 3 \\ \quad - x^3 + 4x^2 - 3x + 3 \\ \hline -2x^6 + 2x^5 - x^4 - 3x^3 \\ \quad + 8x^5 - 8x^4 + 4x^3 + 12x^2 \\ \quad\quad - 6x^4 + 6x^3 - 3x^2 - 9x \\ \quad\quad\quad + 6x^3 - 6x^2 + 3x + 9 \\ \hline -2x^6 + 10x^5 - 15x^4 + 13x^3 + 3x^2 - 6x + 9 \quad \text{Product.} \end{array}$$

(ii) Ascending powers.

$$\begin{array}{l} 3 - 3x + 4x^2 - x^3 \\ 3 + x - 2x^2 + 2x^3 \\ \hline 9 - 9x + 12x^2 - 3x^3 \\ \quad + 3x - 3x^2 + 4x^3 - x^4 \\ \quad\quad - 6x^2 + 6x^3 - 8x^4 + 2x^5 \\ \quad\quad\quad + 6x^3 - 6x^4 + 8x^5 - 2x^6 \\ \hline 9 - 6x + 3x^2 + 13x^3 - 15x^4 + 10x^5 - 2x^6 \quad \text{Product.} \end{array}$$

Multiplication sums can be checked by giving x a numerical value and working the resulting arithmetical sum. In the last example, put $x = 1$.

Then $3-3x+4x^2- x^3 = 3-3+4-1 = 3$,

$3+ x-2x^2+2x^3 = 3+1-2+2 = 4$;

and the product found above $= 9-6+3+13-15+10-2 = 12 = 4 \times 3$.

The results of the following multiplications are so important that they should be learnt by heart.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \quad -b^2 \end{array} \qquad \begin{array}{r} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ -a^2b-ab^2-b^3 \\ \hline a^3 \quad -b^3 \end{array}$$

$$\begin{array}{r} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ +a^2b-ab^2+b^3 \\ \hline a^3 \quad +b^3 \end{array}$$

The formulae to be recollected are therefore:

$$(a+b)(a-b) = a^2-b^2,$$

$$(a-b)(a^2+ab+b^2) = a^3-b^3,$$

$$(a+b)(a^2-ab+b^2) = a^3+b^3.$$

There is no formula giving the factors of a^2+b^2.

75. Sometimes only one term is wanted in a product, e. g. find the term containing x^2 in the product of $3x^3-2x^2+x-4$ and $2x^3-3x^2+4x+5$.

Here it would be labour wasted to work the sum in full. Starting with $2x^3$, it is seen that no term containing x^2 can be obtained from the multiplication by $2x^3$; going to $-3x^2$ it is seen that, in order to obtain a term involving x^2, $-3x^2$ must be multiplied by a mere number, in this case -4. Proceeding in

a similar manner with the other terms, the whole work would appear thus:—

Term involving $x^2 = +12x^2 + 4x^2 - 10x^2$
$= 6x^2$.

Examples X c.

Multiply together:

1. $3x^3 - 2x^2 + x - 2$ and $x - 2$.
2. $4x^4 - 3x^3 + 2x^2 + x - 2$ and $3x - 4$.
3. $5x^3 - 2x^2 + 3x - 1$ and $2x - 5$.
4. $3x^3 - 3x^2 + 2x - 5$ and $x^2 - 2x - 2$.
5. $7x^2 - 3x + 1$ and $3x^2 - 3x + 4$.
6. $3x^3 - 2x^2y + 3xy^2 - 4y^3$ and $x - 2y$.
7. $2a^3 - 2a^2 + 3a - 4$ and $2a^2 - 3a + 3$.
8. $4a^4 - 6a^3 - 2a + 1$ and $3a^3 - 2a - 4$.
9. $a - 2a^3 + 3a^2 - 5$ and $2 - a + a^2$.
10. $3ab + 2bc + 3ac$ by $2a + 3b - 4c$.

Find the term involving x^3 in the products of:

11. $1 + 2x + 3x^2 + 4x^3 + 5x^4$ and $1 - x + x^2 - x^3 + x^4$.
12. $1 - 3x + 5x^2 - 7x^3 + 9x^4$ and $1 - 2x + 3x^2 - 4x^3 + 5x^4$.
13. $2 - 4x + 6x^2 + 8x^3 + 10x^4$ and $3 - 9x + 27x^2 + 81x^3$.
14. $1 - ax + a^2x^2 - a^3x^3 + a^4x^4$ and $2 - 3bx + 4b^2x^2 - 6b^3x^3$.
15. $x^4 - 2x^3 + 3x - 4$ and $x^5 - 43x^4 + x^2 - 2x + 3$.

Find mentally the products of:

16. $x - 2y$ and $x + 2y$.
17. $x + 2y$ and $x^2 - 2xy + 4y^2$.
18. $p - 3q$ and $p^2 + 3pq + 9q^2$.
19. $2a - 3b$ and $2a + 3b$.
20. $2a + 3b$ and $4a^2 - 6ab + 9b^2$.
21. $3x - 5y$ and $9x^2 + 15xy + 25y^2$.
22. $p - 4$ and $p^2 + 4p + 16$.
23. Multiply $3a^3 - 4a^2 + 2a - 7$ by $2a^2 + 3a - 1$.
24. Find the product of $3 - 2x - 3x^2$ and $2 - 3x^2 - 5x^3$.
25. What must be multiplied by $x - 3$ to give a product $x^3 - 8x + 15$?
26. What must be multiplied by $x - 3$ to give a product $x^3 - 27$?
27. Multiply $ax^2 + bx + c$ by $bx^2 - cx - a$.
28. Add the product $(a + b + 1)(a - b + 2)$ to the product
$(a + b - 1)(a - b - 2)$
and verify the result when $a = 4$, $b = 1$.

29. Multiply together (x^2+x+1), x^2-x+1, and $x+1$.

30. Find the product of

$$a(1+b)-1 \text{ and } a^2(1-b+b^2)+a(1+b)+1.$$

Multiply together:

31. $\frac{1}{2}a-\frac{1}{3}b$ and $\frac{1}{2}a+\frac{1}{3}b$.

32. $\frac{1}{2}a+\frac{2}{3}b-\frac{1}{5}c$ and $\frac{1}{5}a-\frac{1}{2}b$.

33. $\frac{2}{3}x^2-\frac{1}{4}x+\frac{2}{3}$ and $\frac{1}{3}x^2-\frac{2}{3}x+\frac{1}{4}$.

34. $2a^2-\frac{1}{3}a+\frac{1}{5}$ and $\frac{1}{2}a^2-\frac{2}{3}a+\frac{1}{2}$.

35. $\frac{a}{3}-\frac{1}{5}$ and $\frac{a^2}{9}+\frac{a}{15}+\frac{1}{25}$.

Examples X.

1. Find the product of the three factors $(x+y)$, $(y+z)$, $(x+z)$. Arrange the answer in descending powers of x.

Test your result by putting $x=-1$, $y=2$, $z=3$ in the factors and in the product.

2. Multiply $4a-3b+5c$ by $-6a^2b^2$.

3. If $P=x^2+2xy+2y^2$, $Q=x^2-2xy+2y^2$, $R=x^4-y^4$, find the value of $PQ-R$.

4. Multiply $5x^3-xy^2-7y^3$ by $x^2-4xy+3y^2$. Verify the result when $x=-2$, $y=1$.

5. What is the product when $(1+x)(1+x^4)$ is multiplied by $(1-x)(1+x^2)$?

6. Multiply $4a^2-2a+1$ by $4a^2+2a+1$.

7. Multiply together $(x+a-b)(x+b-c)(x-a+c)$, collecting in brackets the coefficients of the powers of x.

8. If $2s=a+b+c$, prove that $2s(2s-2a)+(2s-2b)(2s-2c)$ equals $4bc$.

9. Simplify

$$(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)+(x+y+z)^2.$$

10. Find the area of a room whose length is x^2 yards x feet 4 inches and breadth x yards 2 feet x^2 inches.

11. What quantities multiplied together give a product x^2+x-6?

12. Multiply $x^2+2ax+2a^2$ by $x^2-2ax+2a^2$.

13. Multiply x^3-x^2-9x+9 by $4x^3+4x^2-x-1$.

14. Multiply $9x^2+3xy+y^2-6x+2y+4$ by $3x-y+2$.

15. What will a oranges, b apples, c pears cost if the price is $(a+2b-c)$ pence each?

16. Multiply $2-3x$ by $8+12x+18x^2+27x^3$.

17. Multiply $5a^3-3a^2b+2ab^2+5b^3$ by $10a^2+6ab-15b^3$.

18. Multiply $\frac{y}{3}-2+\frac{z}{2}$ by $\frac{z}{2}+2+\frac{y}{3}$.

19. Multiply together $1+x-y$, $1-x+y$, $1-x-y$.

20. What is the product of $a+b+c$ and $a^2+b^2+c^2-ab-ac-bc$?

21. What are the factors of x^2-4x-5?

22. Simplify

$$(a-b)(a+b+2c)+(b-c)(b+c+2a)+(c-a)(c+a+2b).$$

23. Multiply together $y+z-2x$, $z+x-2y$, and $x+y-2z$. Put $x=1$, $y=1$, $z=1$ in the result, and hence show how we could tell beforehand that the algebraical sum of the coefficients in the product would be zero.

24. If $x-3$ will divide x^2-5x+c without remainder, what is the value of c?

25. What is the coefficient of x in the product of $(x-2a)$, $(2x+3)$, $(3x-4)$? What value must a have in order that this coefficient shall be zero?

26. Prove that $a^m \times a^n = a^{m+n}$.

27. Show that $(a+c)(a-c)-b(2a-b) = (a-b+c)(a-b-c)$.

Verify when $a=1$, $b=-1$, $c=0$.

28. Prove that $(a+b)c = ac+bc$ when a, b, and c are positive whole numbers.

Simplify $(2a+3b)(2x-3y)-6(a+x)(b-y)$.

29. Multiply together $9x^2+7x+3$ and $3x^2+7x+9$.

Hence find the product of 973 and 379.

30. Multiply $x+y-1$ by $x^2+y^2+1+x+y-xy$.

31. If a and b stand for any numbers, express in words the fact that $(a+b)(a-b) \equiv a^2-b^2$.

32. Simplify by two different methods $(5x-7)^2-(x-5)^2$.

33. If $2b = a+c$, show that $(a-b)^2+2b^2+(b-c)^2 = a^2+c^2$.

34. Simplify

$$(x+1)(x-2)-2(x+2)(x-3)+(x+3)(x-1).$$

35. Simplify

$$(5x+3y)^2-(4x-5y)^2+(3x-4y)[2(x-y)-(5x+2y)].$$

36. Find the sum of the squares of $a-b+c-d$ and $a-b-c+d$.

37. Multiply $\frac{3}{4}x^3-\frac{1}{2}x^2+\frac{1}{8}$ by $\frac{4}{3}x-2$.

38. If $p = b+c-2a$ and $q = c+a-2b$ show that

$$p^2+q^2-2pq = 9(a-b)^2.$$

39. Simplify $7x^3-(3-4ax)-3x(4x+1)(ax+5)$; and rebracket the result according to descending powers of x.

40. Arrange $a(b^2-c)+b(c^2-a)+c(a^2-b)$ in descending powers of c; and find its value in terms of c when $a=2b=3c$.

Solve the following equations:

41. $(3x-5)(4x-7)-2(3x-1)(2x-1)=\frac{5x+3}{4}$.

42. $4(2x+3)(3x+1)-12(x+1)^2=(4x+5)(3x+1)$.

43. $(x-2)(x-3)-6(x-3)(x-4)+5(x-2)(x-4)=16$.

44. $(x-1)(x-4)-(x-2)(x-5)=(x+1)^2-x(x-3)-37$.

45. $2(x+2)(4x-5)-(4x+1)(2x-3)=8x+3$.

46. $(x+3)(x-1)(2x+5)=(x-4)(x+2)(x-5)$
$+(x+10)(x-2)(x+8)$.

47. $(5x-6)(x+1)-(7-4x)=8(1+x^2)-3(x-2)^2$.

48. $3(x+1)(x+3)-2(x^2-1)=(x-1)^2+3(5x+1)$.

49. Prove the identity
$(a-b+c)(a^2+b^2+c^2+bc-ca+ab)\equiv a^3-b^3+c^3+3abc$.

50. Prove the identity
$(a-b)(x-a-b)+(b-c)(x-b-c)+(c-a)(x-c-a)\equiv 0$.

CHAPTER XI

DIVISION

76. Division is the process that reverses Multiplication, so that 'Divide x by y' is only another way of saying 'What quantity when multiplied by y gives a product x?', or 'What quantity must y be multiplied by to give a product x?'

The quantity to be divided is called the **dividend**, the quantity which divides is called the **divisor**, the result is the **quotient**. Hence dividend = divisor multiplied by quotient.

77. The rule for Division follows at once by reversing the rule for Multiplication.

Rule. **When one power of a quantity is divided by another power of the same quantity, the quotient is a power whose index is obtained by subtracting the**

index of the divisor from the index of the dividend, e. g. $x^7 \div x^3 = x^{7-3} = x^4$.

Example. *Divide* $18\,a^4b^5c^7$ *by* $3\,a^3b^2c^4$.

$$\text{Quotient} = 18\,a^4b^5c^7 \div 3\,a^3b^2c^4$$
$$= \tfrac{18}{3}\,a^{4-3}b^{5-2}c^{7-4}$$
$$= 6\,ab^3c^3.$$

78. A fraction bears the same relation to division that brackets do to multiplication, for $\dfrac{3x^2-5x+7}{2x}$ means $3x^2-5x+7$ divided by $2x$, and each term in the numerator may be divided separately by $2x$.

Example. *Divide* x^3-4x^2+2x *by* $2x$.

$$\text{Quotient} = \frac{8x^3-4x^2+2x}{2x}$$
$$= 4x^2-2x+1.$$

Or a method similar to short division in Arithmetic may be used:

$$2x \mid 8x^3-4x^2+2x$$
$$4x^2-2x+1 \quad \text{Quotient.}$$

79. In Arithmetic there are two ways of expressing the fact that the divisor is not contained exactly in the dividend: $37 \div 5$ gives quotient 7 and remainder 2, or $37 \div 5$ gives quotient $7\frac{2}{5}$. Both methods say that 5 is contained in 37 seven times, and that when the seven fives are taken from 37 there remains 2 still to be divided. Similarly in Algebra: $4x^2-2x+10 \div 2x$ gives quotient $2x-1$ and remainder 10 or

$$(4x^2-2x+10) \div 2x = \frac{4x^2-2x+10}{2x} = 2x-1+\frac{10}{2x}.$$

Examples XI a.

Divide:

1. $6\,a^2b^4$ by $3\,ab^2$.
2. $-9\,a^3b^3c^2$ by $3\,abc$.
3. $21\,x^2y^4z^3$ by $-7\,xy^2z$.
4. $7\,x^3y^2z^3$ by $-x^2yz$.

5. $-36\,a^4b^3x^2y$ by $-4\,a^2bx$.
6. $13\,a^3x^4y^2$ by $13\,ax^3y^2$.
7. $14\,a^3x^5y^7$ by $-2\,a^2xy^2$.
8. $-15\,a^4b^3c^2$ by $-5\,a^3b^2c$.
9. $12\,abc$ by $6\,abc$.
10. $7\,a^4b^2c^3$ by $3\,a^2bc^2$.

Simplify:

11. $\dfrac{6\,x^3-4\,x^2+2\,x}{2\,x}$.
12. $6\,x^3\times 4\,x^2\div 2\,x$.
13. $\dfrac{9\,a^4+27\,a^3-6\,a^2+3\,a}{3\,a}$.
14. $\dfrac{6\,y^4+3\,y^3+9\,y^2+3\,y^5}{-3\,y^2}$.
15. $\dfrac{6\,a^2bc^2-12\,a^3b^4c^5+18\,a^4b^3c^4}{3\,a^2bc^2}$.
16. $3\,x^2\times 2\,y^2z^2\div 6\,xyz$.
17. $\dfrac{1}{2\,x}(4\,x^3-6\,x^2+8\,x)$.
18. $-\dfrac{3\,x^3+3\,x-6\,x^2}{x}$.
19. $\dfrac{x^4-6\,x^3y+12\,x^2y^2-3\,xy^3}{x}$.
20. $\dfrac{7\,x^4-14\,x^3-21\,x^2-7}{7\,x}$.

Find the quotient and remainder when

21. $3\,a^3b^2-6\,a^2b^3+5\,ab$ is divided by $3\,a^2b^2$.
22. $6\,x^3y^4+4\,x^2y^3+8\,xy^2$ is divided by $2\,xy^3$.
23. $4\,a^4-6\,a^3+8\,a^2-10\,a+3$ is divided by $2\,a^2$.
24. $36\,x^3-24\,x^2y+16\,xy^2-y^3$ is divided by $8\,x^2$.
25. $7\,x^4y^2-14\,x^3y^3+35\,x^2y^4-56\,xy^5+13\,y^6$ is divided by $7\,xy$.

80. It was shown in the previous chapter that such multiplication sums as $(x-4)\times(x+2)$ can be done mentally; the product being $x^2+(-4+2)\,x+(-4)\times(2)$, i. e.

$$x^2-2\,x-8.$$

Hence reversing the process

$$(x^2-2\,x-8)\div(x-4)=x+2.$$

In working division by this mental method, we must see whether the result obtained by subtracting the coefficient of x in the divisor from the coefficient of x in the dividend is the same as the result obtained by dividing the constant in the dividend by the constant in the divisor. If these results are not the same, the division is not exact and there is a remainder

Example. *Divide* $x^2-8\,x-14$ *by* $x-10$.

[Mental work

$-8-(-10)=-8+10=+2$; $(-10)\times(+2)=-20$.

$\therefore$ there is a remainder.]

$$x^2-8x-14 = x^2-8x-20+6$$
$$= (x-10)(x+2)+6;$$

$\therefore$ when $x^2-8x-14$ is divided by $x-10$, the quotient is $x+2$ and the remainder is 6.

Or $$\frac{x^2-8x-14}{x-10} = x+2+\frac{6}{x-10}.$$

Examples XI b.

Find the quotient mentally in the following cases:

1. $(x^2-7x+12)\div(x-3)$.
2. $(x^2+7x-18)\div(x+9)$.
3. $(x^2-4x-12)\div(x-6)$.
4. $(y^2+9y+20)\div(y+4)$.
5. $(y^2-16y+48)\div(y-12)$.

Is there any remainder in the following divisions?

6. $(p^2-4p-7)\div(p-5)$.
7. $(a^2+3a-6)\div(a-3)$.
8. $(x^2-3x+4)\div(x-4)$.
9. $(x^2+2x-8)\div(x+4)$.
10. $(y^2-3y+6)\div(y-3)$.

Find the remainder when

11. (a^2-6a+4) is divided by $a-8$; and put $a=8$ in the dividend.

12. a^2-5a-8 is divided by $a-7$; and put $a=7$ in the dividend.

13. x^2+3x-1 is divided by $x+4$; and put $x=-4$ in the dividend.

14. $x^2-2x+15$ is divided by $x+6$; and put $x=-6$ in the dividend.

15. y^2-y-1 is divided by $y-4$; and put $y=4$ in the dividend.

Find mentally the values of:

16. $\frac{a^2-4}{a-2}$.
17. $\frac{a^3-8}{a-2}$.
18. $\frac{x^2-a^2}{x+a}$.
19. $\frac{y^2-16}{y+4}$.
20. $\frac{p^2-1}{p+1}$.
21. $\frac{p^3+1}{p^2-p+1}$.
22. $\frac{x^2-2x+1}{x-1}$.
23. $\frac{x^2-4x+4}{x-2}$
24. $\frac{y^5+a^5}{y+a}$.
25. $\frac{x^4-a^4}{x^2+a^2}$.

81. Division can be done in Algebra by a method exactly similar to Long Division in Arithmetic.

Example.

Divide $x^5+4x^4-2x^3+x^2-4x+6$ *by* x^2-3x+5.

$x^3+7x^2+14x+8$ Quotient.

$$
\begin{array}{l}
x^2-3x+5)\,x^5+4x^4-2x^3+x^2-4x+6 \\
\qquad x^5-3x^4+5x^3 \\
\qquad\qquad 7x^4-\ 7x^3+\ \ x^2 \\
\qquad\qquad 7x^4-21x^3+35x^2 \\
\qquad\qquad\quad +14x^3-34x^2-\ 4x \\
\qquad\qquad\quad +14x^3-42x^2+70x \\
\qquad\qquad\qquad +\ 8x^2-74x+\ 6 \\
\qquad\qquad\qquad +\ 8x^2-24x+40 \\
\qquad\qquad\qquad\quad -50x-34 \quad \text{Remainder.}
\end{array}
$$

Note. Care must be taken that the divisor and dividend are both arranged in ascending powers of x, or both in descending powers. If any power does not occur, a gap should be left where that power would have come. The division must be continued as long as possible, that is, so long as the first term of the divisor will divide the first term of the remainder. Hence the greatest index in the final remainder is always less than that in the divisor.

82. Horner's Method of Long Division. In the work of the last section it is seen that the first term in any remainder is obtained by a number of subtractions, e. g. to obtain $14x^3$, $5x^3$ is subtracted from $-2x^3$, and $-21x^3$ is subtracted from the result. For each subtraction a mental change of sign is made. Horner's method has the following advantages:

(i) The continual mental change of sign is avoided by actually changing the signs at the beginning of the work.

(ii) The work is so arranged that, instead of several subtractions, only one algebraic addition is necessary.

The successive steps of Horner's method should be carefully studied; with a little practice the method will be

found to be not only neater and shorter, but also easier than the method of § 51.

Example.

Divide $x^5+4x^4-2x^3+x^2-4x+6$ *by* x^2-3x+5.

Step 1. Change the sign of all the terms in the divisor except the first and write them down in a vertical column.

Step 2. Obtain the first term in the quotient by dividing the first term in the dividend by the first term in the divisor and place the quotient under the first term of the dividend. The first column is now finished as shown below.

Step 3. Multiply the terms in the divisor by the first term in the quotient, placing each product in its proper column, viz. x^4 under x^4, x^3 under x^5, &c. and in the same line as the corresponding term of the divisor. The work then stands as follows:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4 \\ -5 & \qquad\quad -5x^3 \\ \hline & x^3 \end{array}$$

Step 4. Add up the first incomplete column and divide the result by the first term of the divisor; this gives the second term of the quotient, and being placed in the bottom line completes the second column.

Step 5. The multiplication proceeds as in Step 3 and the work now stands:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4+21x^3 \\ -5 & \qquad\quad -5x^3-35x^2 \\ \hline & x^3+7x^2 \end{array}$$

Step 6. Proceed as in Steps 4 and 5, and the completed work will be:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4+21x^3+42x^2+24x \\ -5 & \qquad\quad -5x^3-35x^2-70x-40 \\ \hline & x^3+7x^2+14x+8 \quad \text{Quotient.} \end{array}$$

Step 7. We cannot divide any further; adding the two remaining columns we obtain the remainder, viz. $-50x-34$.

Horner's method is particularly useful when the divisor contains only two terms.

Divide $x^7-6x^5+4x^3-3x+5$ by $x+3$.

$$\begin{array}{r|l} x & x^7 \quad -6x^5 \quad +4x^3 \quad -3x+5 \\ -3 & \quad -3x^6+9x^5-9x^4+27x^3-93x^2+279x-828 \\ \hline & x^6-3x^5+3x^4-9x^3+31x^2-93x+276 \text{ Quotient} \\ \hline & -823 \text{ Remainder.} \end{array}$$

The following example is important:

Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.

Long Method.

$$\begin{array}{l} x+y+z \,)\, x^3 \;.\;.\;. -3xyz+y^3+z^3 \,(\, x^2-xy-xz+y^2-yz+z^2 \\ \quad x^3+x^2y+x^2z \\ \quad \overline{\quad -x^2y-x^2z \quad -3xyz} \\ \quad\quad -x^2y \quad -xy^2- \quad xyz \\ \quad\quad \overline{\quad -x^2z+xy^2-2xyz} \\ \quad\quad\quad -x^2z \quad - \quad xyz-xz^2 \\ \quad\quad\quad \overline{\quad xy^2- \quad xyz+xz^2+y^3} \\ \quad\quad\quad\quad xy^2 \quad\quad +y^3+y^2z \\ \quad\quad\quad\quad \overline{\quad -xyz+xz^2 \quad -y^2z} \\ \quad\quad\quad\quad\quad -xyz \quad\quad -y^2z-yz^2 \\ \quad\quad\quad\quad\quad \overline{\quad +xz^2 \quad\quad +yz^2+z^3} \\ \quad\quad\quad\quad\quad\quad +xz^2 \quad\quad +yz^2+z^3 \end{array}$$

Horner's Method.

$$\begin{array}{r|l} z & x^3+0x^2y+0x^2z+0xy^2-3xyz+0xz^2+y^3+0y^2z+0yz^2+z^3 \\ -y & \quad -x^2y \quad +xy^2+xyz \quad -y^3+y^2z-yz^2 \\ -z & \quad\quad -x^2z \quad +xyz+xz^2 \quad -y^2z+yz^2-z^2 \\ \hline & x^2-xy-xz+y^2-yz+z^2 \text{ Quotient.} \end{array}$$

Examples XI.

Divide:

1. $3p^5+16p^4-33p^3+14p^2$ by p^2+7p.
2. $x^5+5x^4-9x^3-38x^2-x+6$ by x^3-7x-3.
3. $-4a^2b-3ab^2+5bc$ by $-4b$.
4. $2x^5+22x^4+127x^2-127x+30$ by $x^2+11x-6$.
5. $12-8x+2x^3-7x^2+x^4$ by $2-3x+x^2$.
6. $16x^5-4x^3-8x^2+x+1$ by $4x^2-2x-1$.

7. $1+3x-24x^3+8x^4$ by $2x^2+3x-1$.
8. $8x^5-12x^3+35x^2+16x-7$ by $2x^2+3x-1$.
9. $5x^6-6x^5+1$ by $1-2x+x^2$.
10. $5x^4-4x^3+3x^2+22x+55$ by x^2-3x+5.
11. $14x^5+33x^3+25x^2+51-68x-34x^4$ by $2x^2+3-4x$.
12. $x^4+18x^3-216x-3888$ by $x^2+12x-108$.

---

13. Prove that the sum of the cubes of two numbers divided by their sum equals the sum of their squares diminished by their product.
14. What is the quotient when x^6-y^6 is divided by
$$(x-y)(x^2+xy+y^2)?$$
15. Find the quotient and remainder when
$6x^4-5x^3y+4x^2y^2-5xy^3+6y^4$ is divided by $x^2-2xy-7y^2$.
16. Find an expression such that when it is divided by b^2-ab+a^2 the quotient is $b^2-2ab+a^2$ and the remainder $2a^3(b-a)$.
17. Divide each of the following expressions by $a-b$:
(i) a^2-b^2+a-b; (ii) $a^3-b^3+ab(a-b)$;
(iii) $x(a-b)+y(a-b)$.
18. If $2x^4+x^3-6x^2-ax-2$ is exactly divisible by x^2+2x+1, find the value of a.
19. Divide each of the following expressions by $x-y$:
(i) $a(x-y)+b(x-y)+c(x-y)$;
(ii) $a(x^2-y^2)+b(x^2-2xy+y^2)$;
(iii) $x^3+x^2+x-y^3-y^2-y$.
20. Add together $1+4x-12x^2$, $12x^2-2+6x$, $-10x-13x^2+2$, and $3x-20x^2-4$. Multiply the result by $3-2x$ and divide the product by $1-x+11x^2$.
21. Find the quotient and remainder when x^3+y^3+8 is divided by $x+y+2$.
22. Divide $x^6-4y^4x^4-y^2x^2+4y^6$ by x^3-y.
23. If $x-3$ will divide $x^5-cx+27$ without remainder, what is the value of c?
24. Divide $x^3+3xy+y^3-1$ by $x+y-1$.
25. Simplify $\dfrac{x^4-3x^3+4x^2-x+5}{x^2+2x-3}$.
26. Divide $2x^3+4x^2+8x+18$ by $x+2$.
What is the value of the dividend if $x+2=0$?
27. Divide $x^4-4x^2-12x-9$ by x^2-2x-3.

28. Divide x^6-1 by x^2-x+1.

29. Find the quotient and remainder when
$x^4-3x^3+5x^2+13x-4$ is divided by x^2-2x+1.

30. Simplify $\dfrac{(3x+4y)^2+(4x-3y)^2}{x^2+y^2}$.

31. Solve $a(x-2b)+b(x-2a)=(a-b)^2$.

32. Solve $a(x+a-b)=b(x-a+b)$.

33. Solve $(b+c)(x-a)-(c+a)\ x-b)=(a+b)(x-c)$.

34. Solve $a(x-a)-b(x-b\ =a-b$.

35. Solve $a(x-3b)+2b(x-a+b)=2(a^2+2b^2)$.

36. Prove that $x^m \div x^n = x^{m-n}$ when m and n are positive integers and m is greater than n.

37. Find the remainder when x^3+px^2+qx+r is divided by $x+a$. What does the dividend become when $x+a=0$?

38. Divide
$1-x+y-xy-x^2y^2-x^2y^3+x^3y^2+x^2y^3$ by $1-x+xy-x^2y$.

39. Solve $\left.\begin{aligned} ax-cy &= ab-c^2 \\ cx-by &= c^2-ab \end{aligned}\right\}$.

40. Divide $x^3-ax^2+a(b+c)x-abc-bx^2-cx^2+bcx$ by $x-a$.

41. Solve $\begin{aligned} x-ny &= (1-n^2)a. \\ ax-by &= a^2-b^2. \end{aligned}$

42. Show by division that $\dfrac{1-x}{1+2x} = 1-3x+6x^2-12x^3+\&c.$

43. Solve $bx-ay=2a+b=2x+y$.

44. Divide $x^4-56x+15$ by $x^2+4x+15$ and verify your answer when $x=10$.

45. Solve $a(x-1)+b(x-a)=x(b-1)$.

46. Find what $12x^5-20x^4-13x^3+32x^2-5x-3$ must be divided by in order that the quotient may be $3x^2-5x+2$ and the remainder $4x-1$.

47. Divide
$x^6-2x^5-(a^2-1)x^4+2a^2x^3+a^2(a^2-1)x^2-2a^4x+a^4$ by x^2-2x+1.

48. Prove that $ax^5+bx^4+cx^3+cx^2+bx+a$ is exactly divisible by $x+1$; and is divisible by $x-1$ if $a+b+c=0$.

49. Divide
$3\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}-2(a^3+b^3+c^3)-12abc$
by $b+c-2a$.

50. Divide $x^2y+xy^2-x^3-y^3-3xy+2x+2y-1$ by $x+y-1$.

CHAPTER XII

INVOLUTION AND EVOLUTION

83. **Involution** is the process of finding powers of quantities.

Examples. $(3x^2)^4 = 3x^2 \times 3x^2 \times 3x^2 \times 3x^2$

$= 3^4 x^{2+2+2+2}$

$= 3^4 x^{2\times 4}$

$= 81x^8.$

$(-3x^2)^3 = (-3) \times (-3) \times (-3) \times x^2 \times x^2 \times x^2$

$= +3^2 \times (-3) \times x^{2+2+2}$

$= -3^3 x^{2\times 3}$

$= -27x^6.$

Examples XII a.

Find the value of:

1. $(2x)^3$. 2. $(-2x)^3$. 3. $(-2x)^6$. 4. $(2x)^6$.
5. $(2x^3)^3$. 6. $(-2x^2)^3$. 7. $(-2x^3)^6$. 8. $(2x^2)^6$.
9. $(3a^2b)^4$. 10. $(-3a^2b^2)^3$. 11. $(-3p^2q^3)^5$.
12. $(2p^2)^3 \times (3p^3)^2$. 13. $(-2ab)^5$. 14. $(x^3y^2)^{10}$.
15. $(x^2y)^4 \times (xy^2)^3$. 16. $(-3ab^2c^3)^4$. 17. $(4a^2bc^3)^6$.
18. $(2a^2b)^3 \div (2ab^2)^3$. 19. 3^{20}. 20. 2^{30}.
21. What is the quantity whose square is $49x^2$?
22. What is the quantity whose cube is $-27x^6$?
23. What is the value of $(-1)^{19}$? What is the value of $(-x^3)^{24}$?
24. Find the value of $(2x^p)^2$, $(-2x^p)^3$, and $(x^p)^q$.
25. What is the value of $(-x^a)^b$?

84. The student will now have found out the following rules:

(i) An even power of a negative quantity is positive, an odd power of a negative quantity is negative.

(ii) $(x^p)^q = x^{pq}$ or in words: **When a power is itself raised to a power, the result is a power whose index is the product of the two original indices.**

(iii) To raise a single term to any power, determine the sign of the result by rule i, raise the coefficient to the required power by multiplication, and multiply the index of every factor in the term by the index of the required power.

85. If $x^n = y$, then x is said to be the n^{th} root of y. That is, the n^{th} root of a quantity is a quantity whose n^{th} power equals the first quantity. Thus since $7^2 = 49$, 7 is the second or square root of 49; since $(x^2)^3 = x^6$, x^2 is the third or cube root x^6; similarly as $(2x^2)^4 = 16x^8$, therefore $2x^2$ is the fourth root of $16x^8$.

The symbol $\surd$ is used to denote a root, e. g. $\sqrt[3]{8}$ means cube root of 8, $\sqrt[5]{243x^{10}}$ means the fifth root of $243x^{10}$. The small figure placed before the symbol is called the index or exponent of the root; the symbol itself is called the **radical sign.**

In the case of the square root the index is omitted; and when there is no index mentioned the word root is always taken to mean square root; thus $\sqrt{x}$ is read root x, and means square root of x.

Evolution is the process of finding any root of a quantity.

As evolution is the exact reverse of involution, the rules for evolution are the reverse of the rules for involution.

Examples XII b.

Find the values of the following roots and in each case prove the answer:

1. $\sqrt{4x^2}$.
2. $\sqrt[3]{8y^3}$.
3. $\sqrt[4]{16x^8y^4}$.
4. $\sqrt{\dfrac{a^4b^6}{x^4}}$.
5. $\sqrt[6]{\dfrac{64x^6}{y^{12}}}$.
6. $\sqrt[3]{\dfrac{81a^6b^3}{8\times x^9}}$.
7. $(\sqrt{7})^2$.
8. $\sqrt{-4x^2}$.
9. $\sqrt[3]{-27p^3}$.
10. $\sqrt{\dfrac{4x^{2a}}{y^{4b}}}$.
11. $\sqrt[5]{-243x^{15}}$.
12. $\sqrt{81a^4b^6}$.
13. $(-2)^2 = 4$, what is $\sqrt{4}$?
14. $(-2)^3 = -8$, what is $\sqrt[3]{-8}$?

15. $(+2)^2 = 4$, what is $\sqrt{4}$?
16. $(+2)^3 = 8$, what is $\sqrt{8}$?
17. Find a quantity which when multiplied by itself equals -1.
18. Between what whole numbers does $\sqrt[5]{741}$ lie?

86. From the last set of examples the following rules may be discovered.

(i) An even root of a positive quantity may be either positive or negative; thus $\sqrt{4}$ may be $+2$ or -2; this is usually written $\sqrt{4} = \pm 2$.

An even root of a negative quantity cannot be found, and is said to be an **impossible** or **imaginary** quantity. The only thing known about $\sqrt{-2}$ is that its square is -2.

An odd root of a positive quantity is positive, an odd root of a negative quantity is negative.

(ii) **A root of a power is a power whose index is found by dividing the index of the original power by the index of the root.**

(iii) A root of a number must be found by trial. It is shown in Arithmetic how to find the square and cube roots of any number. It will be shown in a later chapter how any root of a number may be found approximately.

(iv) To find a root of any single term, determine the sign by rule i, the coefficient as suggested in rule iii, and divide the index of every letter factor of the term by the index of the root.

Involution of the Sum or Difference of Two Terms.

87. By actual multiplication it can be shown that

$$(x+y)^2 = x^2 + 2xy + y^2$$

and

$$(x-y)^2 = x^2 - 2xy + y^2.$$

These are formulae and must be learnt by heart. It is useful also to recollect the corresponding rules in full.

The square of the sum of two quantities is equal to the sum of the squares of the quantities increased by twice their product.

The square of the difference of two quantities is equal to the sum of the squares of the quantities diminished by twice their product.

88. The two formulae of the previous section are really only one, and either of them can be regarded as a particular case of the other.

$$(x+y)^2 = x^2 + 2xy + y^2.$$

y may stand for any quantity ; we may therefore replace it by $-y$. The formula then becomes

$$\{x+(-y)\}^2 = x^2 + 2x(-y) + (-y)^2 ;$$

i. e. $$(x-y)^2 = x^2 - 2xy + y^2.$$

89. There are two simple formulae for the cube of the sum or difference of two quantities.

$$(x+y)^3 = (x+y)^2 \times (x+y)$$
$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

$$\begin{array}{l} x^2 + 2xy + y^2 \\ x \quad + \quad y \\ \hline x^3 + 2x^2y + \quad xy^2 \\ \quad + \quad x^2y + 2xy^2 + y^3 \\ \hline x^3 + 3x^2y \; + 3xy^2 + y^3 \end{array}$$

Similarly or by substituting $-y$ instead of y we have $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.

There is another form for the 'cube' formulae which is often more useful.

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$= x^3 + y^3 + 3xy(x+y);$$

and similarly

$$(x-y)^3 = x^3 - y^3 - 3xy(x-y).$$

90. Any higher power of the sum or difference of two quantities must be found by actual multiplication, which can often be shortened by using the index law for multiplication.

Thus $(a-b)^4 = (a-b)^2 (a-b)^2$
$$= (a^2 - 2ab + b^2)^2$$
$$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \quad \text{(see § 91)},$$

$$(2a+3b)^5 = (2a+3b)^2\,(2a+3b)^2\,(2a+3b)$$
$$= (4a^2+12ab+9b^2)^2\,(2a+3b)$$
$$= (16a^4+96a^3b+216a^2b^2+216ab^3 + 81b^4)\,(2a+36) \quad \text{(see § 91)};$$

and the work can now be finished by multiplication.

Examples XII c.

Find the value of

1. $(2x+3)^2$.	2. $(x-3y)^2$.	3. $(3a+2b)^2$.
4. $(2x+3a)^2$.	5. $(3x-2a)^2$.	6. $(3p+4q)^2$.
7. $(x^2-y^2)^2$.	8. $(3x^2+2y^2)^2$.	9. $(4abc-2xy)^2$.
10. $(3x^2+4yz)^2$.	11. $(x-2pq)^2$.	12. $(3ab+5bc)$.
13. $(a-b)^3$.	14. $(a+b)^3$.	15. $(p-2q)^3$.
16. $(2p+q)^3$.	17. $(3p-4a)^3$.	18. $(2x+3y)^3$.
19. $(ab-3)^3$.	20. $(2x-5z)^3$.	21. $(2ab-3xy)^3$.

Find by a graphical method:

22. $(a+b)^2$.	23. $(x+3)^2$.	24. $(2x+3y)^2$.
25. $(x-y)^2$.	26. $(x-4)^2$.	27. $(a+b+c)^2$.

Find, if possible, the square root of

28. $p^2+2pq+q^2$.	29. $4p^2-4pq+q^2$.
30. $4x^2+12xy+9y^2$.	31. $4x^2-12xy+9y^2$.
32. $4a^2-16ab+9b^2$.	33. $9a^2-30ab+25b^2$.
34. $9x^2+14y^2-12xy$.	35. $16x^2+40xy+25y^2$.

What must be added to the following expressions to make them exact squares?

36. a^2+2ab.	37. a^2+b^2.	38. $4x^2-4xy$.
39. x^2-6x.	40. $9x^2+4y^2$.	41. $a^2b^2+2ab^2c$.

Use the formulae for $(a+b)^2$ and $(a-b)^2$ to obtain

42. 101^2.	43. 997^2.	44. 96^2.
45. 57^2.	46. 72^2.	47. 89^2.

Find the value of

48. $(x+y)^4$.	49. $(2x-y)^5$.	50. $(x+2y)^6$.

91. The formulae of § 89 are true for any quantities; we can therefore deduce a formula for the square of the algebraic sum of any number of quantities.

We have $(x+y)^2 = (x^2+2xy+y^2)$.

Put $x=a$, $y=b+c+d$;

then $(a+b+c+d)^2 = a^2+2a(b+c+d)+(b+c+d)^2$. (i)

In exactly the same way it can be shown that

$$(b+c+d)^2 = b^2+2b(c+d)+(c+d)^2$$

Hence substituting in i, we have the formula

$$\begin{aligned}(a+b+c+d)^2 = a^2 &+2a(b+c+d)\\ &+b^2+2b(c+d)\\ &+c^2+2cd+d^2.\end{aligned}$$

Hence we have the rule.

The square of the sum of any number of terms equals the square of the first term + twice the first term multiplied by each of the following terms + the square of the second term + twice the second term multiplied by each of the following terms + the square of the third term + twice the third term multiplied by each of the following terms, and so on till we come to the square of the last term.

Thus $(2x+3y-4z)^2 = 4x^2+12xy-16xz$
$+9y^2-24yz+16z^2.$

Careful attention must be paid to the sign of each of the products.

Examples XII d.

Find the square of each of the following expressions:

1. $a+b-c$.	2. $x-y-z$.	3. $-x-y-z$.
4. $2a+3b+1$.	5. $2x+3y-4z$.	6. $3y-4x+2z$.
7. x^2+x-1.	8. x^2-x-2.	9. $3x^2+2x+5$.
10. $2a+3b-c+2d$.	11. x^3-2x^2+3x+1.	12. $3p+2q+r-5s$.

When $x=(a-b)$ and $y=(a+b)$, find the value of each of the following:

13. $2x^2+3y^2$	14. $3x^2-2y^2$.	15. $4x^2-9y^2+xy$.
16. x^2-y^2.	17. $8x^2+4y^2$.	18. $(3x+2y)^2$.

Find the values of the following expressions when

$$x=a,\ y=(b-c):$$

19. $(3x-2y)^2$. 20. $(x+2y)^2$. 21. $(2x-3y+a-2b)^2$.

22. Simplify $(x+y)^2+(x-y)^2$ and $(x+y)^2-(x-y)^2$.

23. Prove that $(9x-1)^2-2(7x+3)^2$ equals $(x-9)^2-2(3x+7)^2$.

24. If $p=b+c-2a$ and $q=c+a-2b$, show that

$$p^2+q^2-2pq = 9(a-b)^2.$$

Evolution of expressions containing more than one term.

92. It is shown in § 87 that $(a+b)^2 = a^2+2ab+b^2$, hence reversing the operation $\sqrt{a^2+2ab+b^2} = a+b$. Therefore if an expression consists of three terms of which two are exact squares and the third is twice the product of the square roots of the other two, then the square root of the whole expression can be found.

Examples. $16x^2+24xy+9y^2 = (4x)^2+(3y)^2+2(4x)(3y)$

$\therefore \sqrt{16x^2+24xy+9y^2} = 4x+3y.$

$4a^2+25b^2-20ab = (2a)^2+(5b)^2-2(2a)(5b)$

$\therefore \sqrt{4a^2+25b^2-20ab} = 2a-5b.$

The sign $\pm$ should be placed before each square root, but it is often omitted.

93. In a similar way by reversing the rule of § 91 it is sometimes possible to find the square root of expressions containing more than three terms.

Example. Find the square root of

$$4x^2+12xy-16xz+9y^2-24yz+16z^2.$$

We see that there are three square terms, $4x^2$, $9y^2$, $16z^2$. This suggests that the square root is $2x \pm 3y \pm 4z$. Squaring this last expression, we get the right product terms by taking the second term positive and the third term negative.

Hence the square root $= 2x+3y-4z$.

The solution can also be arranged thus:

Expression $= (2x)^2+4x(3y-4z)+(3y-4z)^2$.

$\therefore$ sq. root $= 2x+3y-4z$.

Examples XII e.

Find when possible the square roots of the following expressions:

1. $p^2-2pq+q^2$.	2. $4x^2-4x+1$.	3. $9x^2+12xy+4y^2$.
4. $9x^2+16y^2$.	5. $x^2+2xy-y^2$.	6. $4x^2-8ax+4a^2$.
7. $4a^2-12ab+9b^2$.	8. $y^2-10yz+25z^2$.	9. $49a^2+112ab+64b^2$.
10. a^2+4b^2-4ab.	11. $8ab+4b^2+4a^2$.	12. x^4-10x^2+1.
13. a^4+4+4a^2.	14. $1-2x+x^2$.	15. $z^2-2zx+x^2$.

16. $a^6-6a^3b^3+9b^6$. 17. $p^{10}-18p^5+81$. 18. b^4+b^2+1.
19. $(x+y)^2-2(x+y)(a+b)+(a+b)^2$.
20. $x^2-2xy+y^2+2(x-y)a+a^2$
21. $a^2-2ab+b^2-2ac+c^2+2bc$.
22. $4a^2+9b^2+16c^2+12ab-16ac-24bc$.
23. $x^2+4xy-6xz+4y^2-12yz+9z^2$.
24. $a^2+b^2+c^2+2ab-2bc+2ac$.
25. $4x^4-12x^3+13x^2-6x+1$.
26. $9p^2-6pq+q^2-24p+8q+16$.
27. $x^2+y^2+z^2-2xy-2xz-2yz$.

Examples XII f.

1. If $x=\sqrt{3}$, $y=\sqrt{5}$, what is the value of x^2y^2? What is the value of xy?

2. What is the square of each of the following:
(i) $\sqrt{7}$; (ii) $\sqrt{9\frac{1}{2}}$; (iii) $\sqrt{9017}$.

3. Prove that $\sqrt{x}\times\sqrt{y}=\sqrt{xy}$.

4. Show that $(2-\sqrt{3})^2=7-4\sqrt{3}$.

5. Find the value of $(\sqrt{5}+2)(\sqrt{5}-2)$.

6. Find correct to two decimal places $(\sqrt{17})^2-(\sqrt{15})^2$

7. What must $\sqrt{7}$ be multiplied by to give
(i) 7; (ii) $\sqrt{21}$; (iii) 14?

8. If $x=\dfrac{-1+\sqrt{13}}{2}$, show that $x^2+x-3=0$.

9. If $x=5$ and $y=3$, find the value of
$$x\sqrt{x^2-8y}+y\sqrt{x^2+8y}.$$

10. If $a=3$, $b=-2$, find the value of $\sqrt{a^4-5b^3}$.

11. If $a=1$, $b=3$, $c=-5$, $d=0$, find the value of
$$abc+\frac{c}{ab}+abd+\sqrt{c^2-b^2}.$$

12. If $x=8$, $y=6$, $z=0$, $\omega=4$, find the value of
$$\frac{xy}{4\omega}-\frac{7}{52}xyz^2-\sqrt[3]{\frac{xy^3-z}{\omega^3}}.$$

13. Show that $x=12$ is a root of the equation
$$\sqrt{2x+1}-\sqrt{x+4}=1.$$

14. What is the value of x^2y when $x=3$, $y=2$? Hence show that $\sqrt{18}=3\sqrt{2}$.

15. Show that $\sqrt{50} = 5\sqrt{2}$.

16. Which is the greater $3\sqrt{2}$ or $2\sqrt{5}$?

17. What must $\sqrt{x}-\sqrt{y}$ be multiplied by to give $x-y$ as product?

18. Find the value of $\sqrt{x}$ when x has the following values: 0, 1, 1·21, 1·44, 1·69, 1·96, 2·25. Hence draw the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 2{\cdot}25$. By means of the graph find $\sqrt{1{\cdot}8}$ and verify by calculation.

19. Draw the graph of $y = x^2$ from $y = 0$ to $y = 4$; and use it to find the value of $\sqrt{3}$ and $\sqrt{2}$.

20. Simplify $\dfrac{7-5}{\sqrt{7}+\sqrt{5}} + \dfrac{6-5}{\sqrt{6}-\sqrt{5}}$, and hence find correct to two decimal places the value of $\dfrac{2}{\sqrt{7}+\sqrt{5}} + \dfrac{1}{\sqrt{6}-\sqrt{5}}$.

Examples XII.

1. A regiment can be formed into a solid square with x men in each rank. How many men are there in the regiment? How many additional men are required to form a solid square with 5 more men in each rank?

2. A man who is x years old has a son y years younger than himself. He leaves the son as many pounds for each year of his son's age as there are years in the son's age. How much does the son receive?

3. Prove that the difference of the squares of any two consecutive odd numbers can be divided by 8.

4. A man buys x articles at x pence each. If he spends $4a^2 - 12ab + 9b^2$ pence, what is the value of x?

5. A man has x gold coins and y silver coins. Each gold coin is worth as many shillings as there are silver coins, and each silver coin as many pence as there are gold coins. What is the value of the coins in shillings?

6. The area of a field is $4a^2 + 12ab + 9b^2$ square yds. What is the length of a side?

7. Simplify $(y^2)^n (xy)^{m-n} \div (x^2)^m \left(\dfrac{y}{x}\right)^{m+n}$.

8. Simplify $\left(\dfrac{x^{14}}{x^5}\right)^4 \div \left(\dfrac{x^{13}}{x^6}\right)^5$.

9. Find the square root of $x^{12}(x^6-2)^2+4(x^6-1)^2$.

10. Extract the square root of

$$4x^6+16x^5+28x^4-39x^2-36x+36.$$

11. Extract the square root of

$$6ax(x^3-a^2b)+x^2(x^4-2a^2bx+9a^2)+a^4b^2.$$

12. If $x=3$, $y=-2$, $z=5$, find the value of

$$y\sqrt{z^2-4y^2}-5y^2(3xz-2y^5)-16x^2y^3.$$

13. Simplify $\left(\frac{a^2b}{c^3}\right)^2\times\left(\frac{bc^2}{a^2}\right)^3\div\left(\frac{b+c}{a}\right)^2$.

Find the square root of each of the following expressions:

14. $4x^4-20x^3+13x^2+30x+9$.

15. $16a^4-8a^3-55a^2+14a+49$.

16. $9a^4-12a^3+22a^2-12a+9$.

17. $16x^4-24x^3y-7x^2y^2+12xy^3+4y^4$.

18. $4(x-1)(x^3-1)+9x^2$.

19. $9x^4-6x^3y-11x^2y^2+4xy^3+4y^4$.

20. To the square of double the product of any two consecutive numbers add the square of the sum of the numbers. Prove that the result is always a perfect square.

21. Simplify $(a-b+c)^2+(a+b-c)^2-4(b-c)^2$.

22. Express in algebraical symbols: The difference of the cubes of any two numbers exceeds the cube of their difference by three times their product multiplied by their difference. Prove that the statement is true.

23. Find the sum of $(\overline{a+b}+\overline{c-d})^2$, $(\overline{a+b}-\overline{c-d})^2$, $(\overline{a-b}+\overline{c+d})^2$ and $\{-(a-b)+(c+d)\}^2$.

24. Represent the following statement symbolically:

The n^{th} power of a certain number exceeds the n^{th} root of the number by the q^{th} root of its p^{th} power.

25. If $x=-p\pm\sqrt{p^2-4q}$, find correct to two places of decimals the value of x when

(i) $p=1$, $q=-2$;
(ii) $p=-1$, $q=-2$;
(iii) $p=-2$, $q=-8$;
(iv) $p=5$, $q=3$;
(v) $p=-6$, $q=7$;
(vi) $p=10$, $q=-10$.

26. If a line of length x inches is cut from a line whose length is a inches, what is the length of the remainder?

What is the answer when $a=1$, $x=\dfrac{\sqrt{5}-1}{2}$?

In this case prove that $a(a-x)=x^2$.

CHAPTER XIII

REVISION CHAPTER

94. Definitions. The words term (§ 3) and factor (§ 5) must be carefully distinguished. Note that a term may contain several factors and that a factor may contain several terms, e.g.

$$2ab+3ax-4bc$$

where each term consists of three factors and $(3a+b)(a+3b)$ where each factor consists of two terms.

An expression consisting of one term is called a **monomial** expression, of two terms a **binomial** expression, one of three terms is a **trinomial**, and of more than three a **multinomial** expression.

The definitions of 'power' and 'index' (§ 12) must be known thoroughly. The two are often confused, but it ought to be easy to recollect that the 'power' is the result of the multiplication, and that the 'index' merely shows how many factors are multiplied together. Thus x^3 is the third power of x and equals $1\times x\times x\times x$; the index is 3 and shows that there are three factors equal to x. Each letter factor of a term is called a **dimension**, and the number of dimensions is the **degree** of a term; thus $5a^3b$ which means $5aaab$ is of four dimensions, i.e. is of the fourth degree.

The **degree** of an expression or equation is determined by the degree of the term of highest dimensions in the expression or equation. Thus $2x^3-3x^2+4x+7$ is of the third degree.

A **homogeneous** expression is one in which all the terms are of the same dimensions; the expression

$$3x^4-4x^3y+6x^2y^2-3xy^3+y^4$$

is homogeneous since each term is of the fourth degree.

Equations and expressions of the first degree are often said to be **linear**; if of the second degree they are said to be **quadratic**, if of the third degree **cubic**.

95. Signs of equality and inequality:

$=$ means 'equals'.

$>$ means 'is greater than'.

$<$ means 'is less than'.

If these signs are crossed out they have the negative meaning, thus

$\neq$ means 'is not equal to'.

$\not>$ means 'is not greater than'.

$\not<$ means 'is not less than'.

$\equiv$ is often used to signify 'identically equals' and shows that the equation is an identity.

Note that both $>$ and $<$ point towards the smaller quantity.

96. Positive and Negative Numbers. The whole of Chapter III is important. Note the three ways of regarding negative numbers:

(i) as the result of subtracting positive numbers from nothing;

(ii) as obtained by counting from nothing in a direction opposite to that in which positive numbers are counted;

(iii) as having a meaning exactly opposite to the meaning attached to positive numbers.

These three points of view are quite consistent and all lead to the result that

$$+a+(-a)=0.$$

This may be regarded as the essential property of negative numbers.

The fact that x is positive is often denoted by $x>0$; similarly $x<0$ is another way of saying that x is negative.

97. If a and b are two numbers such that $a-b$ is positive, then a is said to be greater than b, i. e. if $a-b>0$ then $a>b$.

It follows that any positive number is greater than any negative number, e. g. $\cdot 00001 > -1000000$.

Fundamental Laws.*

98. Algebra may be divided into two divisions, Arithmetical Algebra and Symbolical Algebra. In Arithmetical Algebra letters are used, but the letters are supposed to have such values as make

* Sections 98–101 may be omitted at the first reading.

the operations arithmetically intelligible. This is the kind of Algebra we have used in Chapters I and II. In Arithmetical Algebra the letters are supposed to stand for positive numbers, also in such an expression as $x-y$ it is understood that x stands for a greater number than y.

By our knowledge of Arithmetic it is fairly easy to prove the simple laws of Arithmetical Algebra, such as $a+b=b+a$ and $ab=ba$. In Symbolical Algebra these laws are assumed to be true for all values of the letters; by these assumptions extended meanings are found for the operations of addition, multiplication, subtraction, and division. When once these extended meanings have been grasped, there is no need to keep up the distinction between the two kinds of Algebra.

In Chapter IV we considered special cases and so found extended meanings for the simple operations, but it was pointed out that general rules cannot be proved by taking a few numerical examples. We will now show how these same extended meanings can be found by assuming that the laws of Arithmetical Algebra are to be true whatever values the letters may have.

99. Laws of Association, e.g. $a+(b-c)=a+b-c$ and $a-(b-c)=a-b+c$.

Arithmetical Algebra.

a, b, c must all be positive and b must be greater than c.

$a+(b-c)$ means that to a we are to add the excess of b over c. If we have $a+b$ we have added c too much, therefore c must be subtracted.

$$\therefore \quad a+(b-c)=a+b-c. \tag{i}$$

Again $a-(b-c)$ means that from a we are to subtract the excess of b over c. If we have $a-b$ we have subtracted c too much, therefore c must be added.

$$\therefore \quad a-(b-c)=a-b+c. \tag{ii}$$

In exactly the same way we can prove that

$$a+(b+c)=a+b+c \tag{iii}$$

and

$$a-(b+c)=a-b-c. \tag{iv}$$

These four formulae together form the Law of Association for Addition and Subtraction.

(ii) Symbolical Algebra.

Assume the Law of Association to be true for all values of the letters, and accept as true the results this leads to.

e.g. assume $a+(b-c)=a+b-c$

Put $b = 0$, then $a+(-c) = a-c$. (i)

Again, assume $a-(b-c) = a-b+c$.

Put $b = 0$, then $a-(-c) = a+c$. (ii)

Hence, if the Law of Association is always to be true, the result (i) shows that the addition of a negative quantity must be equivalent to the subtraction of the corresponding positive quantity; while result (ii) shows that the subtraction of a negative quantity must be equivalent to the addition of the corresponding positive quantity. These meanings which must, therefore, be attached to the operations of adding and subtracting negative quantities, agree with what we had already found out by experiment in Chapter IV.

They can also be deduced by assuming the Law of Commutation, viz. $a+b = b+a$.

100. Law of commutation for addition and subtraction, i.e. $a+b+c+d = a+c+d+b = +c+d+b+a$, &c.

It is sufficient to discuss the case of two quantities, viz.

$$a+b = b+a$$

Arithmetical Algebra. If a and b are positive integers (i.e. whole numbers), $a+b$ implies that there are two groups of objects, one containing a objects, the other b objects, and that we wish to find what number we should reach if we count the number of

FIG. 25.

objects in the two groups together. It is clear that the number is the same whether we start counting the a group or the b group. Hence, when a and b are positive integers, $a+b = b+a$.

Graphical method. Draw OA a units long and produce to B so that AB is b units long.

If we measure from O the length of OB is $a+b$ units, but if we measure from B the length is $b+a$ units.

But the length of the line is the same whichever way we measure.

$\therefore$ $a+b = b+a$.

This method applies even if a and b are not integers, provided they are both positive.

We can now prove the law for more than two quantities.

$a+b+c+d = a+c+d+b$ changing two letters
$= c+a+d+b$ „ „
$= c+d+a+b$ „ „
$= c+d+b+a$ „ „

Symbolic Algebra. Assume $a+b=b+a$ as true for all values of a and b.

Suppose $b=-x$ where $x>0$ but $<a$;
then $a+(-x)=-x+a$,
but $a = x+(a-x)$ by the Association Law.
$\therefore \quad a+(-x) = -x+\{x+(a-x)\}$
$= -x+x+(a-x)$ by the Association Law
$= a-x.$ (i)

Suppose $b=-x$ where $x>a$ and $a>0$;
then $a = x-(x-a)$ by the Association Law.
$\therefore \quad a+(-x) = -x+a$
$= -x+\{x-(x-a)\}$
$= -x+x-(x-a)$ by the Association Law
$= -(x-a).$ (ii)

Results (i) and (ii) give us the meaning to be attached to the addition of a negative quantity, and agree with the meaning found from the Association Law in § 99 and with the meaning found by experiment in § 22.

Since $a-x=a+(-x)$, and since the Commutation Law has been assumed to be true for all values, it follows that it is also true for combined addition and subtraction, e. g.

$$a+(-b)+(-c)+d = (-c)+d+(-b)+a.$$
$$\therefore \quad a-b-c+d = -c+d-b+a,$$

assumed by (i) above.

101 Law of Commutation for multiplication, i. e. $ab=ba$.

Arithmetical Algebra. This law has been proved, when a and b are positive, by the graphical method in § 11, Ex. 5.

Symbolical Algebra. Assume $ab=ba$ in all cases.

Suppose ab means a multiplied by b,
and that $a=-x$, where x is positive;
then $(-x)b$ means $-x$ multiplied by b
$= (-x)+(-x)+(-x) \ldots$ b terms
$= -x-x-x \ldots$ b terms
$= -bx;$ (i)
but since $ab=ba$ in all cases,
$\therefore \quad (-x)b = b(-x)$,

i. e. b multiplied by $(-x) = (-x)\,b$
$= -bx$ by (i).

Hence we have the meaning to be attached to multiplication by a negative quantity, viz. multiply by the corresponding positive quantity but change the sign of the product.

This, again, agrees with the rule already found in Chapter IV, § 29.

In a similar way when other laws * of Arithmetical Algebra are assumed to be true in all cases, they can be shown to lead to the same interpretation of the meaning of addition, multiplication, &c., of negative quantities as have been found in Chapter IV.

102. Brackets. Be careful to notice that brackets are not necessarily to be removed whenever they occur. When brackets are used to enclose the coefficient of a power of x they are usually to be kept; when any power of x comes inside them it is usually more convenient to remove the brackets.

Example. *Divide* $x^3-(a+b+c)\,x^2+(bc+ca+ab)\,x-abc$ *by* $x-a$.

Keep the brackets.

$$
\begin{array}{l|l}
x-a & x^3-(a+b+c)\,x^2+(bc+ca+ab)\,x-abc \;\big\lfloor\; x^2-(b+c)\,x+bc \\
\multicolumn{2}{l}{\quad x^3-a\quad x^2} \\
\multicolumn{2}{l}{\quad\overline{\qquad\qquad\qquad\qquad\qquad\qquad}} \\
\multicolumn{2}{l}{\quad -(b+c)\,x^2+(bc+ca+ab)\,x} \\
\multicolumn{2}{l}{\quad -(b+c)\,x^2+(\quad +ac+ab)\,x} \\
\multicolumn{2}{l}{\quad\overline{\qquad\qquad\qquad\qquad\qquad\qquad}} \\
\multicolumn{2}{l}{\qquad\qquad +bc\quad x\quad -abc} \\
\multicolumn{2}{l}{\qquad\qquad +bc\quad x\quad -abc} \\
\multicolumn{2}{l}{\qquad\qquad\overline{\qquad\qquad\qquad\qquad}}
\end{array}
$$

Quotient $x^2-(b+c)\,x+bc$.

Examples XIII a.

Revision Examples on Chapters I—V

1. When $x = 2\,y = -2$, prove that

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)-\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)=\frac{(x+y)^2+(x-y)^2}{xy}.$$

* The rule in § 38 is called the Law of Distribution or Multiplication and is true for any number of factors.

2. In the expression

$$(2a+3)x^3-(4a^3-2a)x^2+(a^2-2a+1)x-a^2,$$

find the coefficients of the different powers of a; and determine the numerical value of the whole expression when $a=3$, $x=-5$.

3. If $a=1$, $b=\frac{1}{2}$, $c=0$, $d=-\frac{1}{2}$, find the numerical value of:

(i) $4ab-3bc-2bd-3ad$;

(ii) $a^2(b+c+d)+a(b^2+c^2+d^2)+\frac{1}{a}(b^3+c^3+d^3)$.

4. Simplify

$$3[ax-2\{-(a^4x^4-8)+3x+7x^2+6x^4\}]-6a^2x^2(1-ax);$$

and arrange the answer in ascending powers of x.

5. Simplify

$$-5\{a-3[a-(b-c)]\}+30\{b-(c+a)\}.$$

6. Find the numerical value of

$$x^2(x+1)+x\{3y^2-y(3x-2)\}-y\{2x+y^2-(x-y)\},$$

where $x=4\frac{1}{2}$, $y=5\frac{1}{2}$.

7. Find the sum of

$$-(c-a)x+(a-b)y+(b-c)z, \quad -(a-b)x-(b-c)y+(c-a)z$$

and $\quad (b-c)x+(c-a)y-(a-b)z$;

and calculate what it becomes when

$$a=b+1\tfrac{1}{2}=c+2\tfrac{1}{2}$$

8. Simplify the expressions:

(i) $-7a+9b-\{-(5a+4c)+4a-(4b+c)\}$

(ii) $-(4a+5b)-\{(5b-3a)+8c-(7b-5a)\}$;

and find the difference between them.

9. If $a=0$, $b=2$, $c=4$, $d=8$, find the value of

$$4\sqrt[3]{bc}-3\sqrt[3]{b^3d}+2\sqrt{ab}+\sqrt{bcd}.$$

10. Simplify $(x-y)^2+(x+y)^2-2(x-y)(x+y)$.

11. Find the value of

$$\frac{x^2}{(x-y)(x-z)}+\frac{yz}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)},$$

when $x=2$, $y=1$, $z=-3$.

12. Divide $a^2(b-c)+b^2(a-c)+c^2(b-a)$ by $b-c$.

13. Divide $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$ by $a+b+c$.

14 Simplify $2a-[3b-\{5c-(3a-5b)-(2c-5a)\}+2b]+3c$.

15. Simplify

$$[(x+y)(x-2y)-(x-2y)(x-3y)+(x+3y)(x-y)]\div(x+y).$$

16. If a, b, c be any positive integers, prove that

$$(a+b)c=ac+bc.$$

17. Find the sum of

$$3x^3+(5x^2y-xy^2),\ 3xy^2-(6x^2y-4y^3),\ 2x^2y-(xy^2+y^3),$$

and subtract $3x^2y-2xy^2$ from the result.

18. Express by means of algebraic symbols the following directions:

'Subtract the product of $2a+3b$ and $3a+4b$ from the product of $3a+2b$ and $4a+3b$, and divide the difference by three times the sum of a and b.'

Find the result.

19. Prove as clearly and as simply as you can that

$$5-(-2)=5+2.$$

Apply the same method to prove that $a-(-b)=a+b$.

20. Prove that $a^2+b^2 \not< 2ab$ whatever a and b may be. (Use the fact that all squares are positive.)

Verify in the following cases:

(i) $a=3,\ b=2$; (ii) $a=3,\ b=3$; (iii) $a=3,\ b=-2$.

21. Which of the following expressions are homogeneous, and which is of the highest degree?

(i) $3x^2-x^3+4x-2$;
(ii) $2a^2b-3ab^2+a^3$;
(iii) $4ab-6a^2b-2a^4-b^2$;
(iv) $3a^2b^3-2ab^2+3b^3$;
(v) $2a^3-3b^3+2c^3-4abc$;
(vi) $3x^2-2y^2+4xy$.

22. Divide $abx^3+(ac-bd)x^2-(af+cd)x+df$ by $ax-d$.

23. Divide $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$ by $x+a$.

24. Prove that $a^3\times a^6=a^9$. What is the formula for dividing a power by another power of the same quantity?

Find the simplest forms for:

(i) $x^7\div x^2$; (ii) $x^2\div x^7$; (iii) $x^7\div y^2$; (iv) $3^7\div 3^2$; (v) $6^3\div 3^2$.

25. Explain why $a-(b-c)=a-b+c$.

Hence show that $a-(-c)=a+c$.

26. What do you understand by the terms positive and negative as applied to numbers and algebraical quantities? State and justify the rule for multiplying a positive by a negative quantity.

27. Simplify $4a-\{3a-(2a+1)\}$.

Give reasons to justify each step in the process of removing the brackets.

28. Define a factor, an index, and a homogeneous expression, and give an example of each.

103. The functional notation. As stated in § 43 any expression whose value depends on the value of x is called

a function of x. Just as we use the letters x, y, &c., when we are talking about numbers in general or about some number whose value is unknown, so we use the symbols $f(x)$, $F(x)$, $\phi(x)$ to denote functions of x. Thus, whenever any form of the letter f comes before a bracket, it is generally not to be regarded as a coefficient, but simply as a short way of writing 'function of'.

This functional notation is often used to save the repeated writing of some definite function. Thus, supposing the function $3x^2-5x+5$ were going to occur frequently, then it would save time and space to write $f(x)$; we might, therefore, write: 'Let $f(x)$ represent $3x^2-5x+5$' or 'let $f(x) \equiv 3x^2-5x+5$'. Throughout that piece of work or example $f(x)$ would then have this definite meaning.

Again, if $f(x) \equiv 3x^2-5x+5$,

then $f(a) \equiv 3a^2-5a+5$;

for if x is replaced by a on one side of the identity, the same change must be made on the other side.

Example. *If* $f(x) \equiv 4x^2-3x+2$, *find the values of* $f(3)$, $f(2)$, $f(1)$, $f(0)$, $f(-1)$.

By question $f(x) = 4x^2-3x+2$.

$$\therefore \quad \begin{aligned} f(3) &= 4 \cdot 3^2-3 \cdot 3+2 = 29, \\ f(2) &= 4 \cdot 2^2-3 \cdot 2+2 = 12, \\ f(1) &= 4 \cdot 1^2-3 \cdot 1+2 = 3, \\ f(0) &= 4 \cdot 0^2-3 \cdot 0+2 = 2, \\ f(-1) &= 4 \cdot (-1)^2-3 \cdot (-1)+2 = 9. \end{aligned}$$

104. Graphs. All the sections on graphs should be revised carefully.

Note the three kinds of graphs.

1. Graphs of statistics or of observations taken at long intervals. These graphs should be a succession of straight lines. Intermediate values cannot be found with this kind of graph.

2. Graphs of observations taken at small intervals. If it is found that a small change in x produces a small change in y we are justified in drawing a curve through the various plotted points. If the observations are correct and are correctly plotted,

it is almost always found that the graph is a neat-looking curve with no awkward bends in it. With such a curve intermediate values can be found with considerable accuracy.

3. Graphs of algebraical functions or equations. These are very much the same as the second kind, but are more regular in shape; and the accuracy of values intermediate to the plotted values depends entirely on the skill of the person drawing the graph.

The process of finding intermediate values is called **interpolation.**

105. The connexion between an equation and a graph must be kept clearly in view.

(i) If simultaneous values of x and y satisfy an equation, the corresponding points lie on a straight line or curve which is called **the graph of the equation.**

(ii) If a point lies on a straight line or curve, its co-ordinates satisfy an equation which is called the equation of that straight line or curve.

(iii) If a pair of values of x and y satisfy two distinct equations, the corresponding point will be the intersection of the graphs of the two equations.

(iv) If a point is the intersection of two graphs, its co-ordinates will satisfy the equations of both those graphs.

Note that one equation is sufficient to draw a graph, but that two equations are necessary to fix the position of a point.

106. Formulae. One of the most useful applications of Algebra is seen in the use of formulae. We give here a few which occur in elementary work.

Arithmetic.

P = principal, r = rate per cent, t = time in years, I = interest, A = amount.

Simple Interest $\quad I = \dfrac{Prt}{100}. \qquad A = P\left(1 + \dfrac{rt}{100}\right).$

Compound Interest $\quad A = P\left(1 + \dfrac{r}{100}\right)^t.$

Algebra.

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$
$$a^2 - b^2 = (a+b)(a-b).$$

Mensuration.

Area of rectangle = base × height, i.e. $A = bh$.

Area of triangle = $\frac{1}{2}$ base × height, i.e. $A = \frac{bh}{2}$.

Circumference of circle = 2π* × radius, i.e. $C = 2\pi r$.

Area of circle = π × square of radius, i.e. $A = \pi r^2$.

Surface of cylinder = circumference × height, i.e. $A = 2\pi rh$.

Surface of cone = $\frac{1}{2}$ circumference of base × slant height, i.e.

$$A = \pi r\sqrt{r^2+h^2}.$$

Surface of sphere = $4\pi r^2$.

Volume of cylinder = area of base × height, i.e. $V = \pi r^2 h$.

Volume of cone = $\frac{1}{3}$ area of base × height, i.e. $V = \frac{1}{3}\pi r^2 h$.

Volume of sphere = $\frac{4}{3}\pi r^3$.

Volume of prism = area of base × height, i.e. $V = Ah$.

Volume of pyramid = $\frac{1}{3}$ area of base × height, i.e. $V = \frac{1}{3}Ah$.

If a, b, c are sides of a triangle and $2s = a+b+c$, then area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

Examples XIII b.

Find the value of $f(0)$, $f(1)$, $f(-1)$, $f(3)$, $f(-5)$ in the following cases:

1. When $f(x) = 3x^3 - 4x - 5$.
2. When $f(x) = 2x^3 - 9x + 2$.
3. When $f(x) = x(x-1)(x-2)$.
4. When $f(x) = 3(x+1)(x-3)(x+5)$.
5. When $f(x) = 2 - 3x + 4x^2 - x^3$.
6. Find the value of $f(a-x) + f(a+x)$ when $f(x) = x^2 + x + 1$.
7. Find the value of $\phi(a+x) - \phi(a-x)$ when $\phi(x) = a^2 + ax + x^2$.
8. Find a function of x of the first degree which equals 5 when $x = 1$, and equals 7 when $x = -1$.
9. If $\phi(x) = ax^2 + 2bx + c$, find the value of
$$\phi(x+1) + \phi(x-1) - 2\phi(x).$$
10. If $F(x) = x + 5$, find the value of
$$F(x+2) \times F(x-2) - \{F(x)\}^2.$$

* π is defined as being the number of times the diameter of a circle is contained in the circumference. It is not an exact number. For practical purposes π may be taken as $3\frac{1}{7}$, or more accurately 3·1416.

11. What can you say about $f'(x)$ if $f(x) = f(-x)$?

12. Draw the graph of $y = f(x)$ if $f(x) = 3 - 2x^2$.

13. Draw the graph of $y = x^2$ from $x = 0$ to $x = 3$, and from the graph find the values of $(\cdot 75)^2$, $(2\cdot 6)^2$, $\sqrt{1\cdot 9}$.

14. A man starts walking at noon; he walks at the constant rate of 4 miles per hour and rests 5 minutes at the end of each fourth mile. Draw a graph by means of which the distance he has walked at any time before 4 p.m. can be found. How far has he walked at a quarter-past two?

15. The area of a triangle is 6 sq. ft.; draw a graph to show how the height varies as the base changes from 1 to 12 feet.

16. Taking π as $3\frac{1}{7}$, draw a graph by means of which the circumference of any circle may be found if the radius is not greater than 5 inches.

17. Find the volume of a sphere whose radius is 3·6 centimetres, taking π as 3·14. {The answer should be given correct to two significant figures.}

18. Find the area of the base of a cone 6 feet high if its volume equals the volume of a sphere 4 feet in diameter.

19. A cylindrical jug 8 inches high holds 1 quart of milk; find the radius of its cross-section. {1 c. ft. of water weighs 1000 oz., 1 gallon weighs 10 lb.}

20. Find correct to one decimal place the area of a triangle whose sides are 13, 15, 16 centimetres respectively.

21. A circular lawn of radius 10 yds. is surrounded by a path 6 feet wide; find the area of the path.

22. Find the simple interest on £105 at $4\frac{1}{2}$ % for 3 years.

23. Find the number of years £210 must be invested at simple interest before it amounts to £252; the rate of interest being 4 per cent.

24. Find the area of a triangle whose sides are 13, 12, 5 inches.

25. Find the area of a triangle whose sides are 5 ft. 5 in., 5 ft. 3 in., and 1 ft. 4 in.

26. If $f(x) \equiv ax^2 + bx + c$, $f(1) = -5$, $f(2) = -2$, and $f(-2) = 10$, find the values of a, b, and c.

27. If $\phi(x) \equiv ax^2 + bx + c$, $\phi(2) = 12$, $\phi(3) = 26$, and $\phi(-1) = 6$, find $\phi(x)$.

28. If $f(x)$ contains only odd powers of x, show that

$$f(x) + f(-x) = 0.$$

29. Draw the graphs of

$$y = 3x - 1,\ 2x - 3y = -4, \text{ and } 3x + 4y = 11.$$

Hence show that there are values of x and y which satisfy all three equations.

30. Draw the graph of $y = x^3 + 6x^2 + 3x - 10$. At what points does it cut the axis of x? Hence solve the equation

$$x^3 + 6x^2 + 3x - 10 = 0.$$

31. Plot the points given by the following table, taking 1 inch as the unit, and drawing a small circle round each point plotted.

x	$-\cdot5$	0	$\cdot5$	1	1·5
y	$-\cdot15$	·25	·8	1·35	1·75

32. Make a table giving the values of $x^2 - 3x$ when x has the values -1, 0, 1, 2, 3, 4. Then draw a curve that will show how $x^2 - 3x$ varies while x varies from -1 to 4. Find from your figure the value of $x^2 - 3x$ when $x = 2\frac{1}{2}$, and verify your result by substituting this value of x in the given expression. Find also for what value of x the expression is equal to 2.

Equations.

107. The distinction (§ 55) between an ordinary equation and an identity is important. As a rule ordinary equations have to be solved, that is, the values of x for which they are true have to be found. To verify that the solutions are correct the values found must be substituted in succession on both sides of the original equation (§ 57).

In practice the coefficients in an equation are often decimals. Unless the decimals are equal to very simple fractions they should not be replaced by vulgar fractions; they should be kept as they are, or may be removed by multiplying throughout the equation by some power of 10.

Example I. *Solve* $\cdot7x - \cdot6x + \cdot875 - \cdot3x = \cdot6$

Transpose $\cdot7x - \cdot6x - \cdot3x = \cdot6 - \cdot875$

Collect $-\cdot2x = -\cdot275$

Divide by $-\cdot2$ $x = \dfrac{-\cdot275}{-\cdot2}$

$= 1\cdot375$ *Answer.*

Example II. *Solve* $3{\cdot}23 - {\cdot}7({\cdot}35 - {\cdot}3x) = {\cdot}24(x - 3)$

Remove brackets $3{\cdot}23 - {\cdot}245 + {\cdot}21x = {\cdot}24x - {\cdot}072$

Transpose $\quad {\cdot}21x - {\cdot}24x = -3{\cdot}23 + {\cdot}245 - {\cdot}072$

$\quad = -3{\cdot}302 + {\cdot}245$

Collect $\quad {\cdot}03x = -3{\cdot}057$

Divide by $-{\cdot}03$ $\quad x = \dfrac{3{\cdot}057}{{\cdot}03}$

$\quad = 101{\cdot}9$ *Answer.*

108. We know that $4 \times 0 = 0$ and $5 \times 0 = 0$; therefore

$$4 \times 0 = 5 \times 0.$$

If we now try to divide both sides by 0, we obtain $4 = 5$ which is untrue. Hence we see that **if both sides of an equation are divided by nothing, the result obtained is not necessarily true.**

Now if a common factor contains the unknown quantity we do not know whether the value of that factor is zero or not; we must therefore never divide by a factor containing the unknown quantity unless we are sure that the factor is not zero.

Example I.

Solve $3x(x-2) - 2(x-2) = 3(x-1)(x-2)$.

Notice that $x-2$ is a factor throughout; so that the equation is satisfied if $x-2 = 0$.

Wrong Method.

Divide by $x-2$ $\quad 3x - 2 = 3(x-1)$.

Remove brackets $\quad 3x - 2 = 3x - 3$.

Transpose $\quad 0 = -1$.

And we have not found the value of x and have got an impossible result.

Right Method.

Either $x-2 = 0$, in which case $x = 2$.

Or $\quad x - 2 \neq 0$, and we may divide by it, and the same work as above leads to $0 = -1$.

So we have this result

either $x = 2$ or $0 = -1$.

The first alternative gives us the required value of x.

Example II.

$$3x(x-2)-2(x-2)=2(x-1)(x-2).$$

Either $x-2=0$, in which case $x=2$.

Or $x-2\neq 0$, and we may divide by it.

Divide by $x-2$ $\quad 3x-2=2(x-1)$.

Remove brackets $\quad 3x-2=2x-2$.

Transpose $\quad x=0$.

Hence we have

either $x=2$ or $x=0$.

Both alternatives are possible, so we have two values of x which satisfy the equation.

Verification.

If $x=2$, $\quad$ L. H. S. $=6(2-2)-2(x-2)=0$,

and R. H. S. $=2(2-1)(2-2)\quad=0$.

If $x=0$, $\quad$ L. H. S. $=0\ 0-2)-2(0-2)=4$,

and R. H. S. $=2(0-1)(0-2)\quad=4$.

Hence both solutions are correct.

Problems.

109. In all problems the following points should be noticed:

(i) Read the question very carefully and make quite sure what is given in the question and what is required to be found.

(ii) Recollect that x stands for a number, and that if a concrete quantity is to be found the units must be clearly stated, e. g.

Let $£x$ be the sum required.

Let x be the number of cows.

Let x years be the man's age, &c.

(iii) Do not give up because the equation cannot be found at once; very often half a dozen steps or more are necessary before the equation can be found.

(iv) If the equation is not obvious, write down in symbolic notation the various quantities that are given in the question; it will soon be found that two different symbolic expressions can be found for the same quantity; hence an equation can be written down.

(v) Recollect that as many equations must be found as there are unknown quantities.

(vi) If there are more unknown quantities than there are equa-

tions, it does not follow that the problem cannot be solved; on the contrary it can be solved in a large number of ways, see § 60.

110. Typical Problem questions. We do not intend to work these fully, but give a few hints which should enable the student to solve them without difficulty.

Number and Digit Problems. *A number of two digits is increased by* 45 *when its digits are reversed; what is the number?*

Recollect that the number 68 (for example) means $6 \times 10 + 8$ and that $365 = 3 \times 10^2 + 6 \times 10 + 5$.

Similarly, if a, b, c are the successive digits of a number, the number is $100\,a + 10\,b + c$.

The digits are called the units digit, the tens digit, the hundreds digit, &c.

In this particular question there are two unknowns, viz. the two digits, but we are told only one fact and can obtain only one equation. By the nature of the question, however, the unknown quantities are positive, integral, and less than ten. Hence all the solutions can be found.

Clock sums. *At what time between* 3 *and* 4 *do the two hands of a watch coincide?*

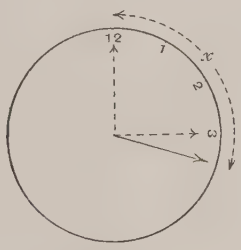


FIG. 26.

Recollect that the minute hand moves twelve times as fast as the hour hand.

Suppose the hands coincide at x minutes past 3.

Then since 3 o'clock the minute hand has moved through x spaces.

But the hour hand has moved through only $x - 15$ spaces.

$\therefore \quad x = 12\,(x - 15)$.

Fraction Problems. *Find a fraction which equals $\frac{1}{3}$ when 1 is subtracted from the numerator, and equals $\frac{1}{2}$ when 1 is added to both numerator and denominator.*

Here there are two unknowns, viz. the numerator and denominator. We are told two facts, therefore two equations can be found.

Coin Problems. *A man has 20 coins in his pocket, some of which are shillings and some are pence; their value is 6s. 3d. How many shillings has he?*

Distinguish carefully between the number of coins and the value of the coins. If two unknowns are used, two equations can be found; one depending on the number of coins, one on their value. In the value equation all the values must be expressed in the same denomination, e.g. if there are x shillings the number of them is x, but their value is $12x$ pence.

Examples XIII c.

Solve:

1. $\frac{5-3x}{4} + \frac{5x}{3} = \frac{3}{2} - \frac{3-5x}{3}$.
2. $\frac{a(a-x)}{b} - \frac{b(b+x)}{a} = x$.
3. $\frac{a(a-x)}{b} - \frac{b(a-x)}{a} = x-a$.
4. $abx = a^2(a-x) - b^2(b+x)$.
5. $\frac{1}{2}(x-\frac{1}{3}) - \frac{1}{3}(x-\frac{1}{4}) + \frac{1}{4}(x-\frac{1}{5}) = \frac{1}{6}$.
6. $\frac{2}{5}x + \frac{15}{2}\left(1 - \frac{x+1}{3}\right) + \frac{19}{10} = \frac{9}{5}\left(\frac{x+1}{6} - \frac{1}{3}\right)$.
7. $4(x+a)(x-b) + 7ab = (2x+a)(2x-b) + (a+b)^2$.
8. $\frac{1}{7}(x+a+2) - \frac{1}{6}(x+2a-6) = 3\frac{1}{3}(x+3a-4)$.
9. $3{\cdot}4x - {\cdot}17(x-2) = 51\left(\frac{x}{5} - 3\right)$.
10. $\cdot3(x-\cdot2) + 3(\cdot7x+5) = 3{\cdot}6\left(\frac{x}{2{\cdot}4} - \frac{1}{2}\right)$.
11. What values of x will make the following statements true?
    (i) $4(x-3) = 5(x-3)$;
    (ii) $6(x-6) = 6 \cdot$

(iii) $x(x-1) = 10(x-1)$;
(iv) $(x-2)(x-7) = (2x-3)(x-7)$;
(v) $(x-5)(x-2) = (x-5)(x-4)$.

12. What values of x will make the following statements true?
(i) $3(x-5) = 4(x-5)$;
(ii) $7(x-2) = x(x-2)$;
(iii) $a(x-b) = a$;
(iv) $(x-1)(3x-4) = (2x-1)(x-1)$;
(v) $(2x-3)(x-2) = (x+12)(2x-3)$.

Solve (and in each case verify the solutions):

13. $7x+9y = x-y = 4$.

14. $\frac{7x}{3} - \frac{5y}{4} = 2\frac{1}{6}$, $4x = 3y$.

15. $ax+by = 2(a^2+b^2)$,
$\frac{x}{a} - \frac{y}{b} = \frac{a}{b} + \frac{b}{a}$.

16. $x+2y = 3a-b$,
$2x+y = 3a+b$.

17. $\frac{4}{x} + \frac{1}{y} = 1$, $\frac{8}{x} - \frac{7}{y} = -\frac{1}{6}$.

18. $\frac{a}{x} + \frac{b}{y} = c+1$, $\frac{ay-bx}{xy} = d-1$.

19. $3x+4y = 11xy$, $5x-2y = xy$.

20. $2x+\frac{1}{2}(6x-2y-1) = \frac{1}{4}+\frac{3}{4}(2y-1)$,
$\frac{1}{5}(4x+3y) = \frac{7}{10}y+1$.

21. $\cdot 12x+\cdot 23y = \cdot 857$,
$1\cdot 4x-3\cdot 1y = -7\cdot 93$.

22. $\frac{\cdot 2}{x} = 1\cdot 3 - \frac{\cdot 45}{x}$.

23. $2x+\cdot 4y = 1\cdot 2$,
$3\cdot 4x-\cdot 02y = \cdot 01$.

24. $\frac{1}{3x} + \frac{1}{5y} = \frac{2}{9}$.
$\frac{1}{5x} + \frac{1}{3y} = \frac{1}{4}$.

25. $3x-7y+4z = 1$,
$-5x+9y-z = 22$.
$x-2y+z = 0$.

26. $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,$

$\frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,$

$\frac{x}{2}+\frac{z}{3}=10.$

27. $xy=3(x+y),\ xz=8(x+z),\ 7yz=9(y+z).$

28. $\frac{1}{3}x-\frac{1}{2}y+\frac{1}{4}z=7,$
$\frac{1}{2}x+\frac{1}{3}y-\frac{1}{3}z=11,$
$\frac{1}{6}x-\frac{1}{4}y+\frac{1}{8}z=5.$

29. $x+8z=2y=z+20=20y-x.$

30. $\frac{1}{2}(x-\frac{1}{2})-\frac{1}{3}(y-\frac{1}{3})=\frac{1}{2}(y-\frac{1}{3})-\frac{1}{3}(x-\frac{1}{2})=\frac{1}{6}.$

31. What do you mean by the term graph? Draw figures giving on any convenient scale the graphs of

(i) $y=-1$; (ii) $y=\frac{x}{2}-2$ from $x=0$ to $x=4$.

Hence obtain the solution of these equations if treated as simultaneous.

Solve graphically and algebraically:

32. $3y=x+2,\ y=3-2x.$

33. $2x+3y-19=0,\ 3x-2y-9=0.$

34. $3x+4y=5,\ 3y-4x=10.$

35. $2y=3(x-4),\ 3y=1-5x.$

36. $5x+3y=7,\ 7y-8x=6.$

37. $2x+y=0,\ \frac{1}{2}y-3x=8.$

38. $y-1\frac{1}{2}=3x-2,\ 2x-1\frac{1}{2}=3-y.$

39. $7x-9y=12,\ 4x+5y=10.$

40. $3x+2y=5,\ 7x-6y=8.$

41. Find the fraction which becomes $\frac{1}{3}$ if 1 be added to its numerator, but, if 1 be added to its denominator, becomes $\frac{1}{4}$.

42. Find that number of two figures to which, if the number formed by changing the places of the digits be added, the sum is 121; and the difference of the same two numbers is 9.

43. A bill of £21. 10*s*. was paid with half-crowns and half-sovereigns, and the number of half-sovereigns exceeded three times the number of half-crowns by 4; how many were there of each?

44. A farmer sells to one person 9 horses and 7 cows for £300, and to another 6 horses and 13 cows for the same sum; find the price of a cow.

45. A purse contains £2. 12*s.* made up of pence, shillings, and half-crowns. The number of shillings is three times the number of half-crowns, but is only one-fourth the number of pence. Find the number of each coin.

46. A bill of £5 is paid partly in English shillings and partly in French francs. If the number of shillings were decreased by one-third, the number of francs would have to be increased by one-half. Find the number of each, assuming that twenty-five francs equal £1.

47. The garrison of a town has provisions for a certain number of days and it is known that, if a thousand men were to leave the garrison, the provisions would last two days longer but that, if a thousand more men were to join it, the provisions would last one day less. Find the number of the garrison.

48. A father is 30 years older than his son ; in 20 years' time he will be double the son's age. Find the present age of the son.

49. A grocer buys eggs at 6*s.* 6*d.* per hundred. He sells all but 69 of them at the rate of 11 for a shilling, and then finds that he has received 30*s.* more than he gave for the whole number. How many eggs did he buy ?

50. A bag contains a certain number of gold coins. Half its contents and half a sovereign more are removed. Half of what remains and half a sovereign more are then removed, leaving £5 in the bag. Find how much the bag contained originally.

51. A boy bought 104 apples for five shillings. The large ones he got at the rate of two for three-halfpence and the small ones two a penny. How many of each sort did he buy ?

52. A train leaves London for Brighton at 9 a.m., travelling at the uniform rate of a miles an hour. Another train leaves Brighton for London at 10 a.m. and travels uniformly at b miles an hour. At what time will they meet, and at what distance from London, the distance between the two towns being taken at 50 miles ?

53. At what time between 8 and 9 o'clock are the hands of a clock in the same straight line ?

54. The dial of a clock shows 24 hours, so that the hour hand makes only one revolution per day and points straight upwards at midnight. At what time between 5 and 6 p.m. do the hands coincide ?

Identities.

111. The two sides of an identity are really the same expression written in different forms, so that an identity is true for all values of x. It is often asked to prove that a given equality is an identity. This should be done by one of the following methods:

(i) Start with either side, and by removing brackets or by other algebraical processes transform it into the other side.

(ii) Take the two sides separately and prove them both equal to some third form.

Example. *Prove that* $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv bc(b-c)+ca(c-a)+ab(a-b)$.

Method I.
$$\begin{aligned} \text{L.H.S.} &= a^2b-a^2c+b^2c-b^2a+c^2a-c^2b \\ &= b^2c-bc^2+c^2a-ca^2+a^2b-ab^2 \\ &= bc(b-c)+ca(c-a)+ab(a-b) \\ &= \text{R.H.S.} \end{aligned}$$

and identity is proved.

Method II.
$$\begin{aligned} \text{L.H.S.} &= a^2b-a^2c+b^2c-b^2a+c^2a-c^2b \\ &= a^2(b-c)-a(b^2-c^2)+bc(b-c) \\ \text{R.H.S.} &= bc(b-c)+c^2a-a^2c+a^2b-ab^2 \\ &= a^2(b-c)-a(b^2-c^2)+bc(b-c). \end{aligned}$$

$\therefore$ L.H.S. = R.H.S.

and identity is proved.

The following very common method should never be used:

$a^2(b-c)+b^2(c-a)+c^2(a-b) = bc(b-c)+ca(c-a)+ab(a-b)$.

Remove brackets:

$a^2b-a^2c+b^2c-b^2a+c^2a-c^2b = b^2c-bc^2+c^2a-ca^2+a^2b-ab^2$.

Transpose and collect: $0 = 0$.

This method is bad: (i) it begins by assuming what has to be proved; (ii) it does not prove what was asked, but proves that $0 = 0$.

112. The following is a common type of example.

If possible, find values of A *and* B *so that*

$$A(x-2)+B(x-3) = 3x-5$$

shall be an identity.

Method I. The equality will be an identity if the coefficients of x on each side are the same, and if the constants are also the same.

$$A(x-2)+B(x-3) \equiv 3x-5,$$

if $$Ax-2A+Bx-3B \equiv 3x-5;$$

i.e. if $$(A+B)x-(2A+3B) \equiv 3x-5;$$

i.e. if $$A+B=3,$$

and $$2A+3B=5,$$

solving $$A=4, \quad B=-1.$$

Method II. If the equality is an identity it will be satisfied by any value of x we like to substitute.

$$A(x-2)+B(x-3) \equiv 3x-5.$$

Put $x=3$. $$A(3-2)+B(0) = 9-5;$$

$$\therefore \quad A=4.$$

Put $x=2$. $$A(2-2)+B(2-3) = 6-5;$$

$$\therefore \quad B=-1.$$

If what is asked is possible, these are the values.

Verify L.H.S. $= 4(x-2)-(x-3) = 3x-5 =$ R.H.S.

The Remainder Theorem.

113. In an Algebraic division sum the final remainder is of lower degree than the divisor; hence, if the divisor is of the first degree in x, the remainder is of no degree in x, i.e. the remainder does not contain x.

Suppose a function of x, viz. $f(x)$, is divided by $x-a$ and that the quotient is Q and the remainder R. Then Q will usually contain x, but R, as has just been shown, will not contain x.

$$\frac{f(x)}{x-a} = Q + \frac{R}{x-a}. \qquad \text{see § 79}$$

This statement is true for all values of x, that is, it is an identity.

Multiply by $x-a$, $$f(x) \equiv Q(x-a)+R.$$

Being true for all values of x, this is true when $x=a$.

Put $x=a$, $$f(a) = 0+R, \text{ since } x-a=0.$$

Rewrite this last result, $$R = f(a) \text{ when } x=a.$$

But since R does not contain x, it will have the same value whatever be the value of x.

$\therefore$ Whatever be the value of x, $R = f(a)$.

Hence when $f(x)$ is divided by $x-a$, the remainder is $f(a)$. This result is so important that it should be recollected in full:

When an expression containing x is divided by $x-a$, the remainder may be found without division by substituting a instead of x in the dividend.

The Factor Theorem.

114. If there is no remainder in a division sum, the divisor is a factor of the dividend. Hence it follows from the remainder theorem that, **if $f(a) = 0$, $x-a$ is a factor of $f(x)$.**

Or, in full:

If, when a is substituted for x in an expression, the result is zero, then $x-a$ is a factor of that expression.

Suppose $x = a$ is a solution of the equation

$$f(x) = 0,$$

then $$f(a) = 0.$$

It follows from the Factor Theorem that if $x = a$ is a solution of $f(x) = 0$, then $x-a$ is a factor of $f(x)$; and conversely, if $x-a$ is a factor of $f(x)$, then $x = a$ is a solution of $f(x) = 0$.

Example I. *Find the remainder when* $x^4 - 2x^3 + x - 3$ *is divided by* $x-2$.

Here
$$f(x) \equiv x^4 - 2x^3 + x - 3$$
$$f(2) = 2^4 - 2 \cdot 2^3 + 2 - 3$$
$$= -1;$$
$$\therefore \text{ required remainder is } -1.$$

Example II. *Are* $x-2$ *and* $x+3$ *factors of* $x^3 - 7x^2 + 13x - 6$?

Here
$$f(x) \equiv x^3 - 7x^2 + 13x - 6,$$
$$f(2) = 2^3 - 7 \cdot 2^2 + 13 \cdot 2 - 6$$
$$= 8 - 28 + 26 - 6$$
$$= 34 - 34$$
$$= 0;$$
$$\therefore \quad x-2 \text{ is a factor.}$$
$$f(-3) = (-3)^3 - 7(-3)^2 + 13(-3) - 6$$
$$= -27 - 63 - 39 - 6$$
$$\neq 0;$$
$$\therefore \quad x-(-3), \text{ i.e. } x+3, \text{ is not a factor.}$$

Examples XIII d.

Find, if possible, values of A, B, C, such that the following equations shall be identities; verify in each case.

1. $A(x-2)+B(x-3)=3x-8$.
2. $A(x-1)+B(x-1)(x-2)=3x^2-7x+4$.
3. $A(x+1)+B(2x-1)=x+4$.
4. $A(2x+1)+B(3x+2)=7x-13$.
5. $A(x-2)+B(x-3)+C(x-4)=3x-4$.
6. $x^2-2x+1=A(x^2+1)+B(x-2)+C$.
7. $(x-1)(x-2)=(Ax+B)(x-3)$.
8. $3x-4=A(2x-3)+B(3x-4)$.
9. $A(x-1)(x-2)+B(x-2)(x-3)+C(x-3)(x-1)=x$.
10. $A(x-1)(x-2)+B(x-2)+C=3x-2$.

Prove the following identities:

11. $(b-c)(x-a)+(c-a)(x-b)+(a-b)(x-c)\equiv 0$.
12. $(x-1)(x-2)+(x-2)(x-3)\equiv 2(x-2)^2$.
13. $(2x-3)^2-(3x-5)^2+(x-2)^2\equiv(2x-3)(4-2x)$.
14. $3(x-2)+4(x-3)\equiv(\frac{1}{2}x-9)^2-(\frac{1}{2}x-10)^2+1$.
15. $(x-1)^2-2(x-1)(x-2)+(x-2)^2\equiv 1$.

Without actual division find the remainder when

16. x^3-3x^2+x-1 is divided by $x-2$.
17. x^4+1 is divided by $x-3$.
18. x^3-5x+4 is divided by $x+1$.
19. $x^4-3x^3+2x^2-3x+1$ is divided by $x-2$.
20. x^5-2x^2-3 is divided by $x+3$.
21. $x^{50}-x+2$ is divided by $x+1$.
22. $x^{48}-2x^{47}+x^9-2x^8+x^3-x^2+4$ is divided by $x-2$
23. $x^4-3x^3+2x^2-1$ is divided by $x-a$.
24. $x^7+6x^6-3x^3+4$ is divided by $x+p$.
25. y^5-3y^2+4y-a is divided by $y-x$.
26. Show that $x-1$ is a factor of
    (i) $x^4-3x^3+5x^2-4x+1$;
    (ii) $3x^2-x^3+2x-4$;
    (iii) $x^{15}-14x^{14}+13$.
27. If $f(x)=x^3+2x^2-5x-6$, find the values of
    $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$.

Hence find the factors of $f(x)$.

By finding the values of $f(x)$ for different values of x, find the factors of $f(x)$ when $f(x)\equiv$

28. x^2+x-2.
29. x^3-3x^2+2x.
30. x^3-4x^2+x+6.

Multiplication and Division.

115. The importance of mental work cannot be too strongly insisted upon. Such sums as $(x+5)(x-2)$, $\frac{x^2-8x+12}{x-2}$, should be worked mentally as explained in § 73 and § 80. Mental work is quite as accurate as written work, saves time and space, and looks neater.

The rules contained in the formula $(a+b)(a-b) = a^2-b^2$ are extremely useful both in Arithmetic and Algebra.

Example I.

$$\begin{aligned}&(3a-4b+5c)(3a+4b-5c)\\&=\{3a-(4b-5c)\}\{3a+(4b-5c)\}\\&=(3a)^2-(4b-5c)^2\\&=9a^2-16b^2+40bc-25c^2.\end{aligned}$$

Example II.

$$\begin{aligned}&(x^4-2x^3+3x^2-4)(x^4+2x^3+3x^2+4)\\&=\{(x^4+3x^2)-(2x^3+4)\}\{(x^4+3x^2)+(2x^3+4)\}\\&=(x^4+3x^2)^2-(2x^3+4)^2\\&=x^8+6x^6+9x^4-4x^6-16x^3-16\\&=x^8+2x^6+9x^4-16x^3-16.\end{aligned}$$

Notice that the terms whose signs are the same in both factors are placed in the first bracket, those whose signs are different in the two factors are placed in the second bracket. Thus in Algebra the formula is used to work out the product of a sum and difference as the difference of two squares; in Arithmetic it is more generally used the reverse way.

Thus, 4934^2-2562^2

$$\begin{aligned}&=(4934+2562)(4934-2562)\\&=7496\times 2372\\&=17780512.\end{aligned}$$

$$\begin{array}{r}7496\\2372\\\hline14992\\22488\\52472\\14992\\\hline17780512\end{array}$$

If this had been worked by squaring, there would have been two long multiplication sums and one subtraction; whereas by this method there are one long multiplication, one subtraction, and one addition.

Detached coefficients.

116. If in the expression $5x^2+3x+4$ we put $x=10$, we obtain $5\times10^2+3\times10+4=534$. We see, therefore, that any arithmetical number can be obtained from an algebraical expression, arranged in descending powers of x, having as coefficients the respective digits that form the number. Hence all arithmetic sums can be worked in the same way as algebra sums, powers of 10 taking the place of powers of x.

Compare the two ways of finding the product of 237 and 569.

Arithmetic.	Algebra.
569	$5\times10^2+6\times10+9$
237	$2\times10^2+3\times10+7$
1138	$10^5+10^4+3\times10^3+8\times10^2$
1707	$10^4+7\times10^3+0\times10^2+7\times10$
3983	$3\times10^3+9\times10^2+8\times10+3$
134853	$10^5+3\times10^4+4\times10^3+8\times10^2+5\times10+3$

i. e. 134853.

It is seen that the Algebraic method is much longer owing to the continual repetition of the tens. The Arithmetic method shows that the index of 10 has been increased by one by placing the coefficient of the power in a column one step further to the left. Exactly the same thing can be done in Algebra, thus enabling us to omit the x's and to put down only the coefficients. It is advisable to write the powers of x at the top of the columns, just as we write £. s. d., or tons, cwt., qr., lb., in Arithmetic.

Multiply $5x^2+6x+9$ *by* $2x^2+3x+7$.

Long method.

$$\begin{array}{r} 5x^2+6x+9 \\ 2x^2+3x+7 \\ \hline 10x^4+12x^3+18x^2 \\ +15x^3+18x^2+27x \\ +35x^2+42x+63 \\ \hline 10x^4+27x^3+71x^2+69x+63 \end{array}$$

Short method.

x^4	x^3	x^2	x	
		5	+ 6	+ 9
		2	+ 3	+ 7
10	+12	+18		
	+15	+18	+27	
		+35	+42	+63
$10x^4$	$+27x^3$	$+71x^2$	$+69x$	$+63$

This short method is called the method of **Detached**

coefficients, and can be applied to addition, subtraction, multiplication, or division.

We will now give an example of the application of detached coefficients to Horner's method of long division.

Divide $5x^4-4x^3+3x^2+22x+55$ by x^2-3x+5.

	x^4	x^3	x^2	x	
1	5	− 4	+ 3	+22	+55
+3		+15	+33	+33	
−5			−25	−55	−55
	$5x^2+11x+11$				Quotient.

Find the remainder when x^5-3x^2-5 is divided by $x-3$.

	x^5	x^4	x^3	x^2	x	
1	1			−3		−5
+3		+3	+9	+27	+72	+216
	$x^4+3x^3+9x^2+24x+72$					

Remainder 211.

Verify by remainder theorem.

Remainder $= 3^5-3^3-5$
$= 243-27-5$
$= 211$

Note. It is often found that the easiest way of calculating $f(a)$ is to find by Horner's method the remainder when $f(x)$ is divided by $x-a$; thus reversing the use of the remainder theorem.

Examples XIII e.

1. Multiply $2a-3b+c$ by $2a+3b+c$.
2. Find the product of $3x^2-2x-1$ and $3x^2+2x+1$.
3. Find as shortly as possible the product of
   x^3+2x^2-x+2 and x^3-2x^2+x+2.
4. Simplify $(x^3+x^2-3x-4)^2-(x^3-x^2-3x-4)^2$.
5. Calculate as shortly as possible 9987^2-9887^2.
6. Find the area of a circular racing track of which the outer radius is 259 ft. and the inner radius 238 ft. ($\pi = \frac{22}{7}$).

In the following examples use the method of detached coefficients:

7. Multiply x^3+2x^2+4x+8 by $x-2$.
8. Multiply x^3+2x^2+3x+4 by x^2-3x+2.

9. Multiply $3x^3+3x^2-x+2$ by $2x^2-3x+1$.

10. Add $3x^2-2ax+5a^2$, $3ax-7a^2+5x^2$, $3a^2-7x^2-4ax$; and subtract $x^2-4ax+a^2$ from the sum.

11. Find the sum of $5x^2-3x-2a$, $4x+3a-3x^2$, $a-2x^2-5x$.

12. Divide $6x^4-19x^3+15x^2+8x-12$ by $2x-3$.

13. Divide $2x^6-3x^5-9x^4+48x+16$ by x^2-4.

14. Multiply x^3-3x^2+3x-1 by $x^2-8x+12$.

15. Divide $2x^5-6x^4+x^3+7x^2-4$ by x^2-3x+2.

16. Divide $x^4+x^3-5x^2+18x-8$ by x^2+3x-2.

17. Divide $6x^5-11x^3+20x^2+23x-38$ by $2x^2-4x+5$.

18. Divide $x^4+18x^3-216x-3888$ by $x^2+12x-108$.

19. Multiply $x^4-2x^3+4x^2-8x+16$ by $x+2$.

20. Divide x^6-6 by x^3-1.

XIII. Revision Examples.

1. Add together $2x+3y-4z$, $5z+5x-4y$, and $7y-3x-8z$; from the sum of these subtract $2x-y-6z$.

2. Multiply x^2+3x+2 by $x-3$, and prove the answer.

3. If $x=3$, $y=0$, $z=-2$, find the value of

(i) $\sqrt{x^2+y^2+4z^2}-\dfrac{x}{z}-\dfrac{8x^3}{z^3}$;

(ii) $z^x-\sqrt[z]{x^2-xz+3z^2}$.

4. Prove that if $a>b>c>0$ then $a-(b-c)=a-b+c$.

5. Solve $\frac{1}{2}(9-2x)=\frac{3}{2}-\frac{1}{10}(7x-18)$.

6. A post is a quarter of its length in the ground, a third of its length in water, and rises 10 feet above the water; what is its whole length?

---

7. Compare the meaning of 59 in Arithmetic with that of ab in Algebra; also the meaning of $5\frac{3}{8}$ and $a\dfrac{b}{c}$.

8. Find the value of $\dfrac{3a^2b^2c}{4a(3bc-4a)}\times(ab+bc+ac)$

when $a=3$, $b=-2$, $c=-4$.

9. Divide $6a^4-13a^3b-4a^2b^2+5ab^3-4b^4$ by $\dfrac{2a^2-3ab-4b^2}{5}$.

10. Show that, if unity be added to the product of any four consecutive numbers, the sum is a perfect square.

11. Solve $\frac{1}{6}(\frac{2}{3}x+4)+\frac{2x-15}{18}=\frac{x}{6}(\frac{6}{x}-1)$.

12. A is twice as old as B; twenty-two years ago he was three times as old. What is A's present age?

---

13. Multiply $ax-(a-2)$ by $ax-(a+2)$.

14. What should be the price of 7 eggs when a dozen cost x pence?

15. Simplify $1-[2+\{3-(1+2)-3\}+1]-2$.

16. Simplify $\frac{a-2}{2}-\frac{a-3}{3}+\frac{a-4}{4}-\frac{a-6}{6}$.

17. Solve (i) $\frac{7x}{3}-\frac{5y}{4}=2\frac{1}{6}$, $4x=3y$;

(ii) $\frac{x-1}{2}+\frac{3x-1}{4}-\frac{5x-1}{6}=\frac{5x-7}{12}$.

18. What is the meaning of $f(x)$?

If $f(x)=x^2-3x+2$, what are the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$?

Draw the graph of $y=x^2-3x+2$ from $x=-1$ to $x=+6$.

---

19. When $a=1$, $b=5$, $c=-1$, find the value of

(i) $(b+c)(c+a)(a+b)$;

(ii) $3abc-a^3-b^3-c^3$.

20. Multiply $x^2+4y^2+9z^2+6yz+3zx-2xy$ by $x+2y-3z$.

21. Find the square root of

$$x^6-6x^4y^2+8x^3y^3+9x^2y^4-24xy^5+16y^6.$$

22. Solve $x-y=3(x+15)$,

$x+y=5(y-9)$.

Verify by a graphical method.

23. A square of carpet is cut up into strips so as to cover a border $(a+b)$ feet wide round the floor of a room whose length is $10a$ feet and breadth $10b$ feet. What was the length of the side of the square of carpet?

24. Prove that $f(a)$ is the remainder when $f(x)$ is divided by $x-a$.

---

25. Subtract $(1+2x)(1+2x-x^2)(1-2x)(1-2x-x^2)$ from the product of $1-2x(1-2x+x^2)$ into $1+2x(1+2x+x^2)$.

26. Write down an expression for the number of which the first digit is a and the second digit b.

Write down also the six numbers that can be formed with the three digits a, b, c in various positions, and find the sum of these six.

27. If a and b be any positive integers, prove that $ab = ba$ and $a+b = b+a$.

28. Given $2y = 5x-6$, find the value of y when $x = 3{\cdot}2$ and when $x = -2{\cdot}4$.

Hence represent graphically the equation $2y = 5x-6$.

What will the equation $2y = 5x-8$ represent?

29. Solve $\left.\begin{aligned}(x-4)(y-3) &= xy-84\\ 6x-5y-27 &= 0\end{aligned}\right\}$.

30. A sum of £5 is divided between two men so that one receives eight times as many pence as the other shillings. What sum does each receive?

---

31. Subtract $\dfrac{3x-6y}{3}+a$ from $x-\dfrac{4y-2c}{2}$.

32. Prove that the square of the sum of any two numbers together with the square of their difference is equal to twice the sum of the squares of the two numbers.

33. Extract the square root of

$$4x^4-12x^5y+17x^4y^2-12x^3y^3+4x^2y^4.$$

34. Solve $\dfrac{x-b}{a} = y-b = \dfrac{x-a}{b}$.

35. Divide $x^5-x^4+8x^2-14x+15$ by x^2-x+3.

36. A man has five sons whose united ages are just equal to his own. In twelve years' time the united ages of the sons will be double that of their father. What is the father's age now?

---

37. Multiply $a^4-3a^3b+4a^2b^2-18ab^3-12b^4$ by $a^2+3ab-2b^2$.

38. Add together $5(a-2b)+b\{3-5(c-2)\}$, $5(b-2)(c-3)$, and $(a+8)(c-4)-(a-2)(c+1)$.

39. A person who has c shillings buys a things each costing b pence; how many pence will he have left?

40. Solve $\dfrac{x-4}{5}+\dfrac{3-x}{6} = \dfrac{5-x}{30}$.

41. What is meant by the graph of a function? Plot the points $(-2, 6)$, $(1, -3)$ and draw the line which joins them. Find its equation.

42. I bought some engravings at 35*s.* each and some books at 16*s.* each. The total cost was £16. 15*s.*, and the number of books was 5 more than the number of engravings. How many were there of each ?

---

43. Prove the identity $(ac-bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2)$.

44. If 1 be divided into any two parts, prove that the results of adding each part to the square of the other are equal.

45. Solve $\cdot 15x+\cdot 2-\cdot 875x+\cdot 375=\cdot 0625x-1$.

46. *A* and *B* begin to play with equal sums, and when *A* has won 30*s.* his money is $\frac{1}{7}^{2}$ of *B*'s ; find how much each started with.

47. Multiply $x^2+4y^2+z^2-2yz+zx+2xy$ by $x-2y-z$.

48. If 8 men do a piece of work in 10 days working x hours a day, and 12 men do the same amount in 8 days working y hours a day, prove that $5x=6y$.

---

49. Divide $x^4-\frac{19}{6}x^2y^2+\frac{1}{3}xy^3+\frac{1}{9}y^4$ by $x^2+2xy+\frac{1}{3}y^2$.

50. Find the values of $(3ax^2z^3)^2$, $(-5a^2bx^3)^3$, $\left(-\frac{4p^3q}{3r^2}\right)^4$.

51. Solve $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.

52. Solve $$\frac{3x+4y+3}{10}-\frac{2x+7-y}{15}=5+\frac{y-8}{5}.$$
$$\frac{9y+5x-8}{12}-\frac{x+y}{4}=\frac{7x+6}{11}.$$

53. A messenger starts on an errand at the rate of 4 miles an hour; another is sent $1\frac{1}{2}$ hours afterwards to overtake him ; the latter walks at the rate of $4\frac{3}{4}$ miles an hour. When will he overtake the first messenger ?

54. Prove the following rule :

To find the price of an oz. avoirdupois when the price per pound in shillings is known. Deduct one-fourth from the price in shillings and the result gives the answer in pence.

---

55. When $a=4$, $b=\frac{1}{2}$, $c=1$, find the value of
$$(a^2+b^2-c^2+2ab)\div(a^2-b^2-c^2+2bc).$$

56. Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$

57. Find the square root of $1+2x+7x^2+6x^3+9x^4$.

58. Solve $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{8}$.

59. Solve graphically and algebraically:

$$\cdot 4x - 2y = 1\cdot 2,$$
$$\cdot 02x + 3\cdot 4y = -\cdot 01.$$

60. The price y in pence of a dish x inches in diameter is given in the following table:

x	10	11	12	13	14	15	16	17
y	$14\frac{1}{2}$	$15\frac{1}{2}$	$16\frac{1}{2}$	$18\frac{1}{2}$	$23\frac{1}{2}$	$30\frac{1}{2}$		51

Draw a graph and obtain an approximate price for the 16 in. dish.

---

61. Prove the following rule: If the square of half the difference of two numbers is subtracted from the square of half the sum, the difference is the product of the numbers.

62. Divide $x^{10} - 123x^5 + 1$ by $x^2 - 3x + 1$.

63. If $a + b = x$ and $a - b = y$, express $a^3 - b^3$ in terms of x and y. Hence find the value of $5002^3 - 4998^3$.

64. Solve (i) $x + y + 39 = 3x - y + 43 = 2y - x + 48$;

(ii) $\frac{1}{3}(x+1) + \frac{1}{4}(5-2x) - \frac{1}{2}(2+5x) + \frac{1}{5}(10-2x) = 0$.

65. A greengrocer has a certain number of apples. He sells one-third of them at three for twopence, two-thirds of the remainder at ten for sixpence, and the rest go bad. On the whole he receives twenty-two shillings. How many apples had he?

66. Draw the graphs $3x - y = 1$, $x + y = 7$. Hence solve the equations and verify by calculation.

---

67. Simplify

$$(a+b)(a^2 - b^2 + c^2) + (c-a)(b^2 - a^2 - c^2) + (a+b-c)(b^2 - c^2 - a^2).$$

68. Find the value of $\dfrac{1}{y-z} + \dfrac{1}{z-x} + \dfrac{1}{x-y}$

when $x = 4\cdot 6$, $y = 3\cdot 4$, $z = 2\cdot 2$.

69. Solve (i) $9(x-9) = \frac{1}{5}(x-9) - (x-9)$;

(ii) $$\frac{2x-3}{15} - \frac{4x-9}{20} = \frac{27 - 5\frac{1}{2}x}{8} - \frac{9}{40}.$$

70. Find the square root of $9x^4 - 6x^3 - 11x^2 + 4x + 4$.

Find correct to two places of decimals the value of $x + y + xy$ when $x = \sqrt{2}$, $y = \sqrt{3}$.

71. What is the distinction in meaning between $a - (b - c)$ and $(a - b) + c$? Prove that they are equal.

72. In a physical experiment the times a body took to fall through certain distances were noted. Plot the following observations on squared paper and draw the graph:

Height of fall, ft.	4·5	4	3	2	1	·5	2·5
Time of fall, secs.	15	14·1	12·2	10	7·1	5	11·2

CHAPTER XIV

FACTORS

117. FACILITY in resolving expressions into factors is so important that we repeat in this chapter the various hints and rules that have already been given. All examples on resolving into factors should be mentally verified; in fact the methods for mental multiplication and for resolving into factors should be studied together.

118. **Case I.** Whenever all the terms of an expression contain the same factor that factor can be placed outside a bracket, and the quotient obtained by dividing by that factor must be placed inside the bracket.

Examples of this are given in Examples V b.

Harder Example. *Find two factors of*

$$(x+a)\,x^2-(x+a)\,(a+b)\,x+(x+a)\,ab.$$

$$\text{Expression} = (x+a)\,x^2-(x+a)\,(a+b)\,x+(x+a)\,ab$$
$$= (x+a)\,\{x^2-(a+b)\,x+ab\}.$$

119. **Case II.** It often happens that one factor will divide part of an expression and another factor the rest of the expression, the two quotients being the same. When this is the case, the factors of the whole expression can be found as shown in the following example.

Example I. *Factorize* $ax^2+2by^2+2bx^2+ay^2$.

It is seen that a divides two of the terms and $2b$ the other two.

Expression $= ax^2 + ay^2 + 2bx^2 + 2by^2$
$= a(x^2 + y^2) + 2b(x^2 + y^2)$
$= (x^2 + y^2)(a + 2b)$.

At once verify mentally to avoid making mistakes.

Example II. *Find the factors of* $x^3 - 3x^2 - x + 3$.

Expression $= x^3 - x - 3x^2 + 3$
$= x(x^2 - 1) - 3(x^2 - 1)$
$= (x - 3)(x^2 - 1)$.

This could also have been arranged as follows:

Expression $= x^3 - 3x^2 - x + 3$
$= x^2(x - 3) - (x - 3)$
$= (x - 3)(x^2 - 1)$.

Examples XIV a.

Resolve each of the following expressions into as many factors as possible:

1. $ax^3 - 2a^2x + 2a^3$.
2. $3x^2y - 4xy^3 + 5xyz$.
3. $6px - 3py + 6pq$.
4. $8ab^2 - 10abc + 12bx^2$.
5. $5x^2 - 15xy + 20y^2$.
6. $x(x - 3) + y(x - 3) - x + 3$.
7. $7a^2(a - 1) - 3ab(a - 1) + b^2(1 - a)$.
8. $xya^2 - 2xyab + xyb^2$.
9. $a^3 - 3a^2 + ay - 3y - a + 3$.
10. $x^2(a + b + c) - xy(a + b + c) + y^2(a + b + c)$.
11. $25x^4y - 40x^3y^2 + 16x^2y^3$.
12. $ac + ab - bc - c^2$.
13. $4(x - 2)^2(x - 3) - (x - 2)^3$.
14. $a^2 + b^2 + 2(ab + ac + bc)$.
15. $ab(c^2 + d^2) + cd(a^2 + b^2)$.
16. $a^2 - 2ab + b^2 - a + b$.
17. $x^2 - a + ax - x$.
18. $a^2 + ab + ac + bc$.
19. $2ax + 3bx - 4ay - 6by$.
20. $6mp - 9mq - 4pn + 6qn$.

By mental work only find the product of:

21. $x - 4$ and $x + 2$.
22. $x + 3$ and $x + 5$.
23. $x - 2y$ and $x - 4y$.
24. $a - 3b$ and $a + 2b$.
25. $a + 2b$ and $a - 2b$.
26. $3x + 4y$ and $3x - 4y$.
27. $a + b + 1$ and $a + b - 1$.
28. $9p + 10q$ and $9p - 10q$.
29. $3k - 5l$ and $3k + 5l$.
30. $x + y + a + b$ and $x + y - a - b$.

120. Case III. By actual multiplication it can be shown that

$$(a + b)(a - b) = a^2 - b^2.$$

This is one of the most important formulae in Algebra and the two corresponding rules should be learnt thoroughly.

The product of the sum and difference of two quantities is equal to the square of the first minus the square of the second.

The difference of the squares of two quantities is equal to the first plus the second multiplied by the first minus the second.

It should be noticed that the quantities may be numbers, single terms or multinomials (§ 94). In applying these rules, the formulae for squaring binomial and multinomial expressions will be continually used, and should, therefore, be revised (§§ 87, 91).

Note. Quadratic binomials of the form x^2+y^2 have no factors.

Examples XIV b.

Find the factors of:

1. $4x^2-y^2$.
2. $9a^2-16b^2$.
3. $16p^2-25q^2$.
4. $9x^2-25a^2$.
5. $64a^2-81x^2$.
6. $49x^2y^2-36a^2b^2$.
7. p^2-121q^2.
8. $10p^2-40x^2$.
9. $12a^2-75b^2$.
10. 100^2-2^2.
11. 79^2-21^2.
12. 84^2-68^2.
13. $(x+y)^2-a^2$.
14. $(a-b)^2-x^2$.
15. $(2a-3b)^2-4a^2$.
16. $(3x+4y)^2-16y^2$.
17. $(3a-6b)^2-81c^2$.
18. $(2a+3b)^2-(a-2b)^2$.
19. $(4x-5y)^2-9z^2$.
20. $(2x-3a)^2-(a-2x)^2$.

Find the product of:

21. $x+2y$ and $x-2y$.
22. $2x-3y$ and $2x+3y$.
23. $x+y+1$ and $x+y-1$.
24. $2x-3y-1$ and $2x+3y-1$.
25. $a+b+c$ and $a-b-c$.
26. $2a-b+3c$ and $2a+b+3c$
27. $a+b+c-d$ and $a-b+c+d$.
28. 198 and 202.
29. 91 and 109.
30. 78 and 82.

Factorize:

31. $a^2+2ab+b^2-c^2$.
32. $4a^2+4ab+b^2-9x^2$.
33. $a^2-2ac+c^2-b^2+2bd-d^2$.
34. $x^2+2xy+y^2+1$.
35. $a^2+6ab+9b^2$.
36. $ax^2+2axy+ay^2-a^3$.
37. $a^2-3a-6b-4b^2$.
38. x^2+x-y^2+y.

39. $a^2-b^2-c^2+d^2-2ad+2bc$.
40. $4x^2-9y^2+25z^2-20xz$
41. $2x(2x+a)-y(y+a)$.
42. $1+2ab-a^2-b^2$.
43. $1-3x-3xy-y^2$.
44. $a^2+2bc-(c^2+2ab)$.
45. x^3-4x.
46. x^4-x^2.

Write down the quotients in the following cases:

47. $(x^2-a^2)\div(x+a)$.
48. $(1-p^2)\div(1+p)$.
49. $(3x^2-3y^2)\div(x-y)$.
50. $(10x^2-10b^2)\div 2(x-b)$.
51. $(27x^2-12y^2)\div(3x+2y)$.
52. $(16x^2-25a^2)\div(4x-5a)$.
53. $\dfrac{25a^2-49b^2}{5a+7b}$.
54. $\dfrac{72p^2-50q^2}{12p-10q}$.
55. $\dfrac{16-49a^2}{7a-4}$.
56. $\dfrac{16-49a^2}{4-7a}$.
57. $\{a^2-(b-c)^2\}\div(a+b-c)$.
58. $\{9a^2-16(b-c)^2\}\div(3a+4b-4c)$.
59. $\{(x+2y)^2-(a-3b)^2\}\div(x+2y-a+3b)$.
60. $\dfrac{(1-x)^2-(3+2x)^2}{4+x}$.

Solve:

61. $(x-a)(a-x)=(b-x)(x-b)$.
62. $a(a-x)=b(x+b)$.
63. $x(a-x)-a^2=x(b-x)-b^2$.
64. $a(a+x)=b(b-x)$.
65. $x(2-x)-b^2=x(b-x)-4$.

121. Case IV. First method. We have seen in § 39 that

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

which is the same as

$$x^2+(a+b)x+ab=(x+a)(x+b).$$

In applying this it must be remembered that a and b need not be positive, and therefore either $a+b$ or ab or both may be negative. The four possible forms are shown by

$$x^2+6x+8=(x+4)(x+2),$$
$$x^2-6x+8=(x-4)(x-2),$$
$$x^2+6x-16=(x+8)(x-2),$$
$$x^2-6x-16=(x-8)(x+2).$$

In all four cases we have to find two numbers whose

product equals the constant term (the term that does not contain x, sometimes called the third term) and whose algebraic sum equals the coefficient of the first power of x.

Careful attention must be paid to the laws of signs. Thus if the third term is positive, the numbers are both positive or both negative;

if the third term is negative, the numbers are of unlike signs;

if the second term is positive, the numerically greater number is positive;

if the second term is negative, the numerically greater number is negative.

Examples XIV c.

Find the factors of:

1. x^2+5x+6.	2. x^2+6x+5.	3. x^2-6x+5.
4. x^2+6x+9.	5. $a^2-7ab+10b^2$.	6. $a^2+7a+12$.
7. $p^2+9p+18$.	8. $y^2-12y+32$.	9. $y^2+12y+35$.
10. x^2-6x+9.	11. $y^2-7y+10$.	12. a^2-6a+5.
13. x^2-6x-7.	14. $y^2-7xy-18x^2$.	15. $a^2-6a-108$.
16. $x^2-10xy+16y^2$.	17. x^2+9x+8.	18. $x^2+3x-10$.
19. $x^2+5x-14$.	20. $a^2+ab-2b^2$.	21. a^2-16b^2.
22. y^2-6y.	23. $x^2-9xy-10y^2$.	24. $12-4x+x^2$.
25. $12-4x-x^2$.	26. $x^2-16x+48$.	27. $x^2-13xy+36y^2$.
28. $a^2-20a+36$.	29. $a^2+12a+36$.	30. $a^2-37a+36$.
31. $a^2-5ax-36x^2$.	32. $y^2+9y-36$.	33. $y^2-16y+64$.
34. $x^2-35x-36$.	35. $y^2+5y-36$.	36. p^2-36.

37. $x^2-(p-q)x-pq$.
38. a^2-a-42.
39. a^2+a-56.
40. x^2-x-72.
41. $x^2-xy-132y^2$.
42. $x^2-x-600$.
43. x^2-x-12.
44. x^2+x-20.
45. $1+3x-2x^2$.
46. x^3-3x^2+2x.
47. $x^2+2x-15$.
48. $x^2-7x-30$.
49. k^2-3k+2.
50. $2x^2-ax-a^2$.
51. $(x-2)^2+x^2-5x+6$.
52. $(x-3)^2+(x-3)(x-4)$.
53. $(x+4)+x^2-3x-28$.
54. $16-6x-x^2$.

Simplify the following fractions:

55. $\dfrac{x^2-5x+6}{x-2}$.
56. $\dfrac{a^2-17a+72}{a-9}$.

57. $\dfrac{a^2-17a+72}{a^2-14a+45}$.

58. $\dfrac{x^2-ax-56a^2}{x+7a}$.

59. $\dfrac{y^2-16xy-17x^2}{17x-y}$.

60. $\dfrac{y^2-24xy+63x^2}{y^2+15xy-54x^2}$.

122. Case IV. Second method. If the constant term is large it is not easy to find the factors ; in such cases the following method may be used.

$$\begin{aligned} x^2-2x-63 &= x^2-2x+1-64 \\ &= (x-1)^2-8^2 \\ &= (x-1+8)(x-1-8) \\ &= (x+7)(x-9). \end{aligned}$$

It is seen that a number is added which, together with the first two terms, will make an exact square ; in order that the expression may be unaltered in value, the number added must also be subtracted ; if the resulting number is also an exact square, the original expression is reduced to the difference of two squares and can be factorized by Case III.

123. Case V. First method. To factorize expression of the form ax^2+bx+c.

We have to find two expressions, $px+q$ and $rx+s$, such that

$$(px+q)(rx+s) \equiv ax^2+bx+c.$$

Comparing the two sides it is seen that

$$\begin{aligned} pr &= a, \\ qs &= c, \\ ps+rq &= b. \end{aligned}$$

Examples. *To factorize* $20x^2+9x-18$.

From the earlier part of this section we know that we have to find two numbers whose product is 20 and two others whose product is -18, such that the sum of the cross products is **9**.

The factors of 20 are 20×1, 10×2, 5×4.

The factors of 18 are 18×1, 9×2, 6×3.

Since the third term is -18 the constant terms in the factors must be of different signs, so that the algebraical sum of the cross products is an actual difference.

$(20x+18)(x-1)$ gives -2 as coefficient of x, (i)

$(20x+1)(x-18)$ gives -359 as coefficient of x, (ii)

$(10x-9)(2x+2)$ gives $+2$ as coefficient of x, (iii)

$(5x+6)(4x-3)$ gives $+9$ as coefficient of x; (iv)

and these are therefore the required factors.

There are nine possible arrangements of the factors of 20 and 18, and allowing for the differences of sign there are eighteen possible coefficients of x. But it is not necessary to try all these, as combinations like i and iii are obviously not the correct factors; for i has a numerical factor 2 and iii has also a factor 2, while the original expression has not got a numerical factor (if it had it should be removed at once and put outside a bracket as in Case I). Also number ii and similar arrangements need not be considered as they obviously give too big a coefficient for x.

The actual work is done mentally, but the working of the correct pair should be put down at the side to show how the factors are obtained.

The work should be set down somewhat as follows:

$$20x^2+9x-18=(5x+6)(4x-3) \qquad \begin{matrix} 20\times1 & 10\times2 & 5\times4 \\ 18\times1 & 9\times2 & 6\times3 \\ 6\times4 & -5\times3 & =9. \end{matrix}$$

This method is complicated when the coefficient of x^2 and the third term have several pairs of factors, as in the example worked above. When they each have only one or two pairs of factors, or when they involve letters and not numbers, the method is quite simple.

$$39x^2+371x-58=(13x-2)(3x+29) \qquad \begin{matrix} 39\times1 & 13\times3 & \\ 58\times1 & 29\times2 & \\ 13\times9- & 2\times3 & =371. \end{matrix}$$

$$abx^2+(a^2-b^2)x-ab=(ax-b)(bx+a) \qquad \begin{matrix} a\times b \\ b\times a. \end{matrix}$$

124. Case V. Second method. With numerical coefficients the following method often takes less time as it does not require so much trial work:

$$20x^2-9x-18=\frac{20^2x^2-9\times20x-18\times20}{20} \qquad \begin{matrix} 18\times\ 20 \\ 6\times\ 60 \\ 12\times\ 30 \\ 3\times120 \\ 15\times\ 24 \end{matrix}$$

$$=\frac{(20x+15)(20x-24)}{20}$$

$$=(4x+3)(5x-6).$$

Step 1. Multiply and divide by the coefficient of x^2.

Step 2. Treat $20x$ as a single quantity; we then have Case iv, since the numerator is of the form $y^2 - 9y - 18 \times 20$. Find factors of 18×20 whose difference is 9; the trial work is shown at the side.

Step 3. Cancel any factors common to the numerator and denominator. It will be found that the denominator always reduces to unity.

In the following examples expressions of the form ax^2+bx+c should at first be factorized by both methods. Experience will teach which is the better method to use in particular cases.

Examples XIV d.

Find the factors of:

1. $x^2-42x+432$.
2. $x^2-16x-377$.
3. $x^2+12x-748$.
4. $9x^2-27x-2736$.
5. $2x^2-x-3$.
6. $ap^2-apq-72aq^2$.
7. $36x^2-42x+12$.
8. $12a^2+7ab-12b^2$.
9. $5y^2-2y-7$.
10. $10a^2-9a-7$.
11. $10x^2-13x-9$.
12. $15a^2+32a+9$.
13. $4x^2-15xy-4y^2$.
14. $18x^2+53x-35$.
15. $12x^2+11x-15$.
16. $2x^2+3x-35$.
17. $2x^2+x-15$.
18. $3x^2-5x-2$.
19. $3x^2+5xy-2y^2$.
20. $12x^2-x-20$.
21. $14x^2-93x+13$.
22. $6x^2-49x+65$.
23. $8x^2-22x+15$.
24. $abx-(a^2-b^2)x+ab$.
25. $4abx^2-6(a^2+b^2)x+9ab$.
26. $6apx^2-(4a^2-9b^2)xy-6apy^2$.
27. $ax^2-(3ab+2)x+6b$.
28. $3a^2x^2-5abx-2b^2$.
29. $10pqx^2-(5p^2+2q^2)x+pq$.
30. $6a^2x^2-15axy+9y^2$.
31. $9x^2+14x-8$.
32. $30x^2-45x-30$.
33. $ax^2-(9a+2)x+18$.
34. $12ax^2-(4a^2-36)xy-12ay^2$.
35. $4x^2+x-3$.
36. $6x^2-19x+15$.
37. $27x^2-48x+16$.
38. $6x^2-7xy-5y^2$.
39. $p(x^2-1)-x(p^2-1)$.
40. $2a^2+3ab-2b^2$.
41. $6x^2-5x-6$.
42. $6x^2-7x-20$.
43. $ab(x^2+y^2)+xy(a^2+b^2)$.
44. $8x^2+13x-6$.
45. $16x^2-24x+9$.
46. $12x^2-23x-24$.
47. $(a^2-1)x^2+(a^2+1)x+a^2-1$.
48. $(a^2-1)x^2-4ax+1-a^2$.
49. $(a^2+1)(x^2-1)-a^2(a^2+2)x$.
50. $(a^2+1)x^2+a^4x+a^2-1$.

125. Case VI. To resolve the sum or difference of two cubes into factors.

It was shown in § 74 by actual multiplication that

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

and $$a^3+b^3=(a+b)(a^2-ab+b^2);$$

or in words—

The difference of the cubes of two quantities equals the first quantity minus the second multiplied by the sum of the squares of the quantities increased by their product.

The sum of the cubes of two quantities equals the sum of the two quantities multiplied by the sum of the squares of the quantities diminished by their product.

As in the case of the difference of two squares the quantities may be numbers, single algebraical terms, or multinomial expressions, e.g.

$$512-125x^3=8^3-(5x)^3$$
$$=(8-5x)(64+40x+25x^2).$$

$$(a+b)^3+(x+y)^3=(a+b+x+y)\{(a+b)^2-(a+b)(x+y)+(x+y)^2\}.$$

126. Case VII. The division sum worked at the end of § 82 provides the means for resolving into factors expressions of the form $x^3+y^3+z^3-3xyz$.

It is there shown that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$
or $$(x+y+z)(x^2-xy-xz+y^2-yz+z^2).$$

In using this formula or any other formula containing the third power the student must recollect that $(-y)^3=-y^3$.

Find the factors of $x^3-8y^3+27+18xy$.

Expression $=x^3+(-2y)^3+3^3-3x(-2y)\times 3$
$$=(x-2y+3)(x^2+2xy-3x+4y^2+6y+9).$$

It is worthy of notice that

$$x^2+y^2+z^2-yz-zx-xy=\tfrac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}.$$

Examples XIV e.

Find the factors of:

1. $8-64y^3$.
2. $1-216x^3$.
3. $125+8x^3$.
4. $8a^3+27b^3$.
5. $343-x^3y^3$.
6. x^3-8.

7. $108a^3-500$.
8. $27y^3+1000z^3$.
9. $a^3-b^3+c^3+3abc$.
10. $a^3-b^3-c^3-3abc$.
11. $a^3-b^3+1+3ab$.
12. $a^3+b^3-1-3ab$.
13. x^3+8y^3.
14. $24a^3-3b^3$.
15. $8x^3-y^3$.
16. $16x^3-54y^3$.
17. $27x^3-64y^3$.
18. $64x^3-125$.
19. $8x^3+27y^3-1+18xy$.
20. $64-8x^3+125y^3+120xy$.
21. a^3-64ab^2.
22. $x^3+y^3+3xy(x+y)$.
23. $a^3x^3-64b^3x^3$.
24. $3pqr-p^3-q^3-r^3$.

What is the quotient when

25. $x^3+3xy+y^3-1$ is divided by $x+y-1$?
26. $a^3-6ab-b^3-8$ is divided by $a-b-2$?
27. $8x^3+125y^3-8+60xy$ is divided by $2x+5y-2$?
28. What is the value of $x^3+3xyz+y^3-z^3$ when $x=42$, $y=61$, and $z=103$?
29. Divide $2(x^3+y^3+z^3-3xyz)$ by $(y-z)^2+(z-x)^2+(x-y)^2$.
30. If $2x+3y=5z$, prove that $8x^3+27y^3-125z^3+90xyz=0$.
31. Find the factors of $(x-y)^3+(y-z)^3$.
32. What value of x will make $(x-7)^3+(x-3)^3$ equal to nothing?
33. Show that $(2a+3b-4c)^3-(3a-2b+c)^3$ is equal to nothing if $a-5b+5c=0$.

Expressions of higher degree than the second.

127. There is no general rule for resolving into factors expressions which contain the third or higher powers. The methods already mentioned are often effective, and the following examples show how they can be used.

By using the facts that $x^4=(x^2)^2$, $x^6=(x^3)^2$, &c., expressions of higher degree can often be reduced to one of the forms already given.

$$\begin{aligned} x^4-y^4 &= (x^2)^2-(y^2)^2 \\ &= (x^2+y^2)(x^2-y^2) && \text{Case III} \\ &= (x^2+y^2)(x+y)(x-y). \\ x^4+7x^2-18 &= (x^2)^2+7(x^2)-18 \\ &= (x^2+9)(x^2-2). && \text{Case IV.} \\ x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 && \text{Case III.} \\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

This result is important, and it is advisable to recollect it as a separate formula.

$$\begin{aligned} x^4-17x^2y^2+16y^4 &= x^4+8x^2y^2+16y^4-25x^2y^2 \\ &= (x^2+4y^2)^2-(5xy)^2 \\ &= (x^2-5xy+4y^2)(x^2+5xy+4y^2) \qquad \text{Case III.} \\ &= (x-y)(x-4y)(x+y)(x+4y). \qquad \text{Case IV} \end{aligned}$$

This last example could have been resolved rather more easily by the method for Case IV :

$$\begin{aligned} x^4-17x^2y^2+16y^4 &= (x^2)^2-17(x^2)(y^2)+16(y^2)^2 \\ &= (x^2-y^2)(x^2-16y^2) \\ &= (x-y)(x+y)(x-4y)(x+4y). \end{aligned}$$

$$\begin{aligned} x^3-3x^2y-3xy^2+y^3 &= x^3+y^3-3x^2y-3xy^2 \\ &= (x+y)(x^2-xy+y^2)-3xy(x+y) \qquad \text{Case II.} \\ &= (x+y)(x^2-xy+y^2-3xy) \\ &= (x+y)(x^2-4xy+y^2). \end{aligned}$$

Notice the resemblance of this expression to the expanded forms of $(x+y)^3$ and $(x-y)^3$, and also notice the differences.

$$\begin{aligned} x^6-y^6 &= (x^3)^2-(y^3)^2 \\ &= (x^3+y^3)(x^3-y^3) \qquad \text{Case III.} \\ &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2); \qquad \text{Case VI.} \end{aligned}$$

or

$$\begin{aligned} x^6-y^6 &= (x^2)^3-(y^2)^3 \\ &= (x^2-y^2)(x^4+x^2y^2+y^4) \qquad \text{Case VI.} \\ &= (x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

128. Intelligent use of the Factor Theorem (§ 114) is often useful.

Consider $x^3-3x^2-10x+24$.

If there is a factor of the form $x-a$, a must be a factor of 24; therefore possible values for a are 1, 2, 3, 4, 6, 12, 24, or -1, -2, -3, -4, -6, -12, -24.

Denoting the original expression by $f(x)$, we have

$f(1) = 1-3-10+24 \neq 0$; $\therefore$ $x-1$ is not a factor.

$f(-1) = -1-3+10+24 \neq 0$; $\therefore$ $x+1$,,

$f(2) = 8-12-20+24 = 0$; $\therefore$ $x-2$ is a factor.

The other factor may be obtained by division.

$$\begin{array}{c|l} 1 & 1-3-10+24 \\ 2 & \quad +2-\ 2-24 \\ \hline & x^2-x-12 \end{array}$$

The other factor.

This last is Case IV.

Hence $f(x) = (x-2)(x-4)(x+3)$.

Instead of division the following process may be used when we are certain of one factor.

$$\begin{aligned} f(x) &= x^2(x-2) - x(x-2) - 12(x-2). \\ &= (x-2)(x^2-x-12) \\ &= (x-2)(x-4)(x+3). \end{aligned}$$

Here we begin by writing $x^2(x-2)$. This has taken into account x^3, but accounts for $-2x^2$ only; hence there is left $-x^2$. Therefore there must follow $-x(x-2)$, for this makes the x^2 term correct. But we have now brought in a term $+2x$; so, in order to make $-10x$, $12x$ must be subtracted. Hence we finish with $-12(x-2)$; this makes the correct coefficient for x, and at the same time brings in the right constant, viz. $+24$. With a little practice this method can be used very quickly, and it looks neater than the division sum.

Instead of dividing by the factor $x-2$, the other factors could be found by continued use of the factor theorem. The objection to this is that sometimes the remaining quadratic factor cannot be resolved into factors of the form $x-a$, and so the time given to trying the factor theorem would be wasted, e. g.

$$f(x) \equiv x^3+5x^2+10x+8.$$

a may be ± 1, ± 2, ± 4, ± 8.

$f(1) = 1+5+10+8 \neq 0$, $\therefore$ $x-1$ is not a factor;

$f(-1) = -1+5-10+8 \neq 0$, $\therefore$ $x+1$ is not a factor;

$f(-2) = -8+20-20+8 = 0$, $\therefore$ $x+2$ is a factor;

$$\begin{aligned} \therefore f(x) &= x^2(x+2)+3x(x+2)+4(x+2) \\ &= (x+2)(x^2+3x+4); \end{aligned}$$

and it is found, by using the second method for Case IV, that x^2+3x+4 has no factors for

$$\begin{aligned} x^2+3x+4 &= x^2+3x+(\tfrac{3}{2})^2-(\tfrac{3}{2})^2+4 \\ &= (x+\tfrac{3}{2})^2+\tfrac{7}{4}. \end{aligned}$$

This is the sum of two squares, and therefore has no real factors.

129. It is understood when finding factors that the coefficients and constants in the factors are to be real and rational. The meaning of this can be seen from the following examples, where it is supposed that there is no restriction.

$$\begin{aligned} x^2-7 &= x^2-(\sqrt{7})^2 \\ &= (x+\sqrt{7})(x-\sqrt{7}). \end{aligned} \qquad \text{Case III.}$$

Here $\sqrt{7}$ can be found to any required number of places by ordinary arithmetic, so that the factors are quite *real*, but $\sqrt{7}$

cannot be found exactly, and is therefore not *rational*. Hence, because 7 is not an exact square, x^2-7 has no rational factors.

$$\begin{aligned} x^2+7 &= x^2-(-7) \\ &= x^2-(\sqrt{-7})^2 \\ &= (x+\sqrt{-7})(x-\sqrt{-7}). \end{aligned}$$

But all real numbers, whether positive or negative, are positive when squared, therefore $\sqrt{-7}$ is not a real number. Hence x^2+7 has no real factors.

Examples XIV f.

Resolve into as many factors as possible:

1. $4x^4-9a^4$.
2. $25x^4-16y^4$.
3. $16a^4-81b^4$.
4. x^4+y^4.
5. $1-625a^4$.
6. $16-49k^4$.
7. x^4-5x^2-6.
8. $x^4+2x^2y^2+9y^4$.
9. $x^4-15x^2y^2+9y^4$.
10. $16x^4+36x^2y^2+81y^4$.
11. $a^4-a^2b^2+b^4$.
12. ax^3-bx^3+bx-a.
13. $20x^3-28x^2-15x+21$.
14. $12a^3+16a^2-3ax-4x$.
15. $x^3-6x^2+11x-6$.
16. $x^4-3x^3+4x^2-6x+4$.
17. x^3-x^2-x+1.
18. x^3-3x+2.
19. $15+19x-4x^2$.
20. x^3-3x^2-x+3.
21. x^4-10x^2+9.
22. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
23. $x^3+3x^2-4x-12$.
24. $x^3+x^2+x-y^3-y^2-y$.
25. $(a-b)^3+(b-c)^3+(c-a)^3$.
26. $x^3+5x^2-4x-20$.

What values of x will make the following expressions respectively equal to nothing?

27. x^2-256.
28. $x^2-13x-30$.
29. $(x-3)(x^2-x-30)$.
30. x^3-4x^2+x+6.
31. x^3-343.
32. x^4-81.

33. If $x+2$ is a factor of $x^4-2x^3-30x^2-47x-6$, what are the remaining factors?

34. Being given that $x+6$ is one factor, find the other factors of

$$x^4+5x^3-20x^2-60x+144.$$

Examples XIV.

Find the factors of:

1. $(x-2)(2x^2-7x+6)+(x-2)(2x-3)$.
2. x^2+x-20
3. $6x^2-7x+2$.
4. $12x^2-17x+6$.
5. $x^3-2x^2y+2xy^2-y^3$.
6. $3x^2+5x-2$.

7. $x^2-(y-2z)^2$.
8. $(a+2b)^3+(a-2b)^3$.
9. $12x^2y^2-17xy+6$.
10. $x^4+2x^3+2x^2+2x+1$.
11. $4x^2-5x+1$.
12. $ax-ab-bx+b^2$
13. $x^4+x^3-12x^2$.
14. $4x^2-9$.
15. $x^3-2x^2y-4xy^2+8y^3$
16. $16x^4-20x^2+4$.
17. $a^2-b^2+2ac+c^2$.
18. x^3-16xy^2.
19. $ad+bx+ax+bd$.
20. $6x^2-ax-2a^2$.
21. $ac+2by+2ay+bc$.
22. $3x^3-8x^2-x+6$.
23. $2x^3-9x^2+4x+12$.
24. $15(x-y)(x+y)-16xy$.
25. $(a^4-b^4)\{c^2-(d-1)^2\}$.
26. $x^4-10x^2y^2+9y^4$.
27. $a-bc-b(1-c)$.
28. $10x^2+13x-6$.
29. $15x^2-13x+2$.
30. $(a+2b-3c)^2-(3a-b+2c)^2$.
31. x^2-x-42.
32. $9a^2+6ab-16c^2-8bc$.
33. $(3a+2b+c)^2-(a+2b+3c)^2$.
34. $3x^3-x^2-6x+2$.
35. x^4+x^2+1.
36. $bc(b-c)+ca(c-a)+ab(a-b)$.
37. $x^2-(y+1)^2$.
38. $(x^2-y^2)^2-2(x^2+y^2)+1$.
39. $x^3-22x-15$.
40. $(a+c)(a-c)-b(2a-b)$.

Find in as few steps as possible the following products:

41. $(x-1)(x+1)(x^2+1)(x^4+1)$.
42. $(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.
43. $(x-2)(x-3)(x-1)(x-6)$.
44. $(2x+1)(4x^2+2x+1)(4x^2-2x+1)(2x-1)$.
45. $(x-3)(x-4)(x-1)(x-6)$.
46. $(x-a)(x-b)(x^2+ab)$.
47. $(x-2a)(2x-a)(2x+2a)(x+a)$.

Find an expression that will exactly divide both

48. x^2-3x+2 and $x^2-8x+12$.
49. x^2-3x+2 and x^3-x-6.
50. $10x^2-x-21$ and $14x^2-11x-15$.
51. $3x^2+14x-5$ and $3x^3+8x^2-1$.
52. $a^2-2ab+b^2+a-b$ and $a^2-2ab+b^2+a^2-b^2$.

Find values of x that will make the following expressions respectively equal to nothing.

53. $5(x-3)-4(x-3)$.
54. x^2-8x-9.
55. $x^2-13x-48$.
56. $3x^2+14x-5$.
57. $6x^2+11x-10$.
58. $x^3-2x^2-21x-18$
59. $x^3-x^2-30x+72$.

Prove that the following expressions have no real factors:

60. x^2+2x+7.
61. x^2-4x+9.
62. $3x^2-6x+5$.
63. $4x^2+3x+2$.

Resolve the following expressions into factors, which may in some cases be irrational:

64. x^2-2x-7. 65. x^2-3x-5.

66. $3x^2-8x+5$. 67. $4x^2-9x+4$.

68. $5x^2+2x-7$. 69. $7x^2-12x+4$.

Resolve into factors:

70. $\frac{1}{a^2}-\frac{7}{ab}-\frac{18}{b^2}$. 71. $\frac{x^3}{y^3}-\frac{z^3}{8}$.

72. $6(3x-y)^2-5(3x-y)(a-2b)+(a-2b)^2$.

73. $x^6+64y^6+12x^2y^2-1$. 74. $8x^3+\frac{343}{x^6}$.

75. $(2a-5)^4+(2a-5)^2(3a+4)^2+(3a+4)^4$.

CHAPTER XV

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

130. A common factor of two or more expressions is a factor which divides each of them. Thus a and b and ab are, each of them, common factors of the two expressions abc and bcd; $(x+a)$ and (x^2+a^2) are common factors of $(x+a)(x+b)(x^2+a^2)$ and $(x+a)(x+c)(x^2+a^2)$.

The common factor which is of highest dimensions is called the highest common factor, and is usually denoted by the capital letters H. C. F.

A common multiple of two or more expressions is a quantity which can be exactly divided by each of those expressions. Thus a^2b^2c is a common multiple of abc and ab^2; x^4-a^4 is a common multiple of $(x+a)$ and $(x-a)$.

The common multiple which is of lowest dimensions is called the lowest common multiple, and is usually denoted by the capital letters L. C. M. The L. C. M. of abc and ab^2 is therefore ab^2c, and not a^2b^2c; the L.C.M. of $(x+a)$ and $(x-a)$ is x^2-a^2.

Numbers have no dimensions, so when dealing with numbers we use the adjectives 'greatest' and 'least' instead of 'highest' and 'lowest', and talk about 'greatest common measure' (G. C. M.), 'least common multiple' (L. C. M.), instead of 'highest common factor', 'lowest common multiple.'

In all respects H. C. F. and L. C. M. serve the same purposes in Algebra as G. C. M. and L. C. M. do in Arithmetic; and the methods for obtaining them are the same in principle as the Arithmetical methods for finding G. C. M. and L. C. M.

Examples XV a.

1. Resolve 30 and 140 into factors; write down all the common factors and the G. C. M.

2. Write down all the common factors of a^2b^2x and abx^2. Hence find the H. C. F.

3. Find the H. C. F. of $30a^2b^2x$ and $140abx^2$.

4. What expressions are common factors of $x(x^2-7x+12)$ and $x^2(x^2+7x-30)$? Find the H. C. F.

5. Find the H. C. F. of $15x(x^2-7x+12)$ and $27x^2(x^2+7x-30)$.

6. Find three common multiples of 16 and 24. What is the L. C. M.?

7. Write down three common multiples of a^2bc^2 and ab^4c^3. Find their L. C. M.

8. Find the L. C. M. of $16a^2bc^2$ and $24ab^4c^3$.

9. Write down three common multiples of x^2-4 and x^2-5x+6. Find the L. C. M.

10. Find the L. C. M. of $15(x^2-4)$ and $21(x^2-5x+6)$.

131. A slight difficulty may have been noticed in some of the preceding examples. Suppose it were required to find the H. C. F. of $16xy^2z$ and $24x^2yz^2$; the H. C. F. of xy^2z and x^2yz^2 is xyz so that $4xyz$, $2xyz$, $8xyz$ are all common factors of $16xy^2z$ and $24x^2yz^2$, and they are of the same dimensions. In such cases it is understood that the coefficient of the required H. C. F. is the G. C. M. of the coefficients of the various expressions; so in this particular example the H. C. F. is $8xyz$.

Similarly, the coefficient of the L. C. M is the L. C. M. of the coefficients of the various expressions.

132. The methods of finding the H. C. F. and L. C. M. can be seen from the following examples.

Example I. *Find the* H. C. F. *of* $3a^2x^3$, $9ax^4y^2$, $6a^2x^2y^2$.

The G. C. M. of the coefficients is 3.

The only letters common to all three are a and x.

$\therefore$ the H. C. F. must contain some powers of a and x.

One of the expressions, viz. the second, contains only the first power of a, $\therefore$ the H. C. F. can contain only the first power of a.

The lowest power of x in the three expressions is x^2, i. e. the highest power of x that is common to all of them is x^2, i. e. the H. C. F. is $3ax^2$.

Similar examples could be written out at length in the same way, but the whole work is usually done mentally.

Example II. *Find the* L. C. M. *of* $3a^2x^3$, $9ax^4y^2$, $6a^2x^2y^2$.

The L. C. M. of the coefficients is 36.

In order that the L. C. M. may be divided by a^2x^3 it cannot contain lower powers than a^2x^3.

In order that the L. C. M. may be divided by ax^4y^2 there must be y^2 and another x in addition to a^2x^3, i. e. there must be $a^2x^4y^2$, and we see that $a^2x^4y^2$ can be divided by $a^2x^2y^2$.

Hence the L. C. M. is $36a^2x^4y^2$.

The work is usually done mentally.

Example III. *Find the* H. C. F. *and* L. C. M. *of*

$$x^2-x-56 \text{ and } x^2+10x+21.$$

$$x^2-x-56=(x-8)(x+7),$$
$$x^2+10x+21=(x+3)(x+7);$$
$$\therefore \text{ H. C. F.}=(x+7),$$
$$\text{and L. C. M.}=(x+3)(x+7)(x-8).$$

Example IV. *Find the* H. C. F. *and* L. C. M. *of*

$$x^2-4x+4,\ x^2-10x+16,\ x^2+x-6.$$

$$x^2-4x+4=(x-2)^2,$$
$$x^2-10x+16=(x-2)(x-8),$$
$$x^2+x-6=(x-2)(x+3);$$
$$\text{H. C. F.}=(x-2),$$
$$\text{L. C. M.}=(x-2)^2(x+3)(x-8).$$

Example V. *Find the* H. C. F. *and* L. C. M. *of*

$$2x^3-11x^2+20x-12 \text{ and } 3x^3-7x^2-8x+20.$$

Both expressions must be factorized; and the Factor Theorem should be used.

By comparing the two coefficients of x^3, it is seen that if there is a common factor the first coefficient must be unity.

By comparing the two constant terms it is seen that the constant of any common factor must be ± 1, ± 2, ± 4. This is all done mentally so far; the work should be set down as follows:

For first expression $f(1) = 22 - 23 \neq 0$,

$$f(-1) = -2 - 11 - 20 - 12 \neq 0,$$

$$f(2) = 16 - 44 + 40 - 12 = 0.$$

$\therefore$ $x-2$ is a factor.

$$\begin{aligned} 2x^3 - 11x^2 + 20x - 12 &= 2x^2(x-2) - 7x(x-2) + 6(x-2) \\ &= (x-2)(2x^2 - 7x + 6) \\ &= (x-2)(2x-3)(x-2) \\ &= (x-2)^2(2x-3). \end{aligned}$$

Hence the only possible common factor is $x-2$ or $(x-2)^2$; try $x-2$.

$$\begin{aligned} 3x^3 - 7x^2 - 8x + 20 &= 3x^2(x-2) - x(x-2) - 10(x-2) \\ &= (x-2)(3x^2 - x - 10) \\ &= (x-2)^2(3x+5). \end{aligned}$$

Hence H. C. F. $= (x-2)^2$,

L. C. M. $= (x-2)^2(2x-3)(3x+5)$.

Examples XV.

Find the H. C. F. and L. C. M. of:

1. $3a^2bc^2$, $6ab^3c^4$, $8a^2bc^3$, $2a^3c^2$.
2. $21x^3y^4z^5$, $12x^4y^3z^2$, $15x^5y^8z^3$.
3. $50a^2x^2y^2$, $20a^3xy^3$, $25a^2x^3y^4$.
4. $5x^3(x-2)$, $10x^2(x-3)$, $16x^5(x^3-5x+6)$.
5. $3x^2(x+1)$, $21x^4(x^2+2x+1)$, $9x^3(x^2-1)$.
6. $a^4(x^2-1)$, $3a^2(x^3-1)$, $6a^5(x^4-1)$.
7. $3a(x^2+2x-8)$, $a^2(x^2-x-20)$, $6a^3(x^2-3x-28)$.
8. $4a^2bc$, $5ac^2x$, $6cx^2y$, $7x^2y^2z^3$.
9. x^2-3x+2, $x^2+10x-11$, x^3-1.
10. $x^2-25x+144$, $x^2-9x-112$.

Find the H. C. F. of:

11. $3x^2+x-2$ and $3x^2+4x-4$.
12. $2x^2+7x-15$ and $x^2+9x+20$.
13. x^3-2x-3 and x^3-x^2-7x+3.

14. x^2-3x+2 and x^3-5x+2.
15. x^2-x-2 and $x^4+x^3-5x^2+x-6$.
16. x^3-2x^2+x-2 and x^4+4x^2+3.
17. x^3+x^2-x-1 and x^3-3x-2.
18. $6x^2+19x-7$ and $10x^3+21x^2-47x+7$.
19. x^3-4x^2+5x-2 and x^3-x^2-x-2.
20. $3x^3+4x^2-x-2$ and x^3-x^2-5x-3.

Find the L. C. M. of:

21. $4a(a+b)$ and $6ab(a^2-ab-2b^2)$.
22. a^3-a^2b, a^2b+ab^2, a^2b-b^3.
23. $x^3+ax-2a^2$, $x^3-2ax+a^2$, $2x^3-ax-a^2$.
24. x^3-1 and x^5+1.
25. $2x^2+7ax-15a^2$, $2x^2+11ax-21a^2$, and $5x^3+59ax^2+163a^2x-35a^3$.
26. $2x^3-x-6$, $3x^3-7x+2$, $6x^3+7x-3$.
27. x^2-1, x^2-6x-7, x^3-3x^2+2x, $3x^2-7x+2$.
28. x^4-x^2, x^4-x, x^4+x^2+1.
29. $2x^3+x-3$, x^3+2x-3, x^3+3x^2-x-3.
30. $6y^2-y-2$, $3y^2+5y+2$, $3y^3-y^2+y+2$.

CHAPTER XVI

FRACTIONS

133. Definition. When a quantity p is divided by another quantity q, the result may be written $\frac{p}{q}$. When the division cannot be performed exactly, this form is called a fraction. The quantities p and q may be either numbers or algebraical expressions; thus

$$\frac{3}{5}, \quad \frac{2x}{3}, \quad \frac{x+a}{3x-a}, \quad \frac{x^2-ax+a^2}{x^2+ax+a^2}$$

are all fractions.

In Arithmetic $\frac{3}{5}$ is regarded as being three times the result of dividing unity by five, and is called three-fifths; hence the lower number gives the name to the parts, in this case 'fifths', and is called the denominator; the upper number shows the number of

parts to be taken, and is called the numerator. These names, 'numerator' and 'denominator', are used in Algebra also to denote the dividend and divisor respectively.

An integral expression is an expression containing no fractions with algebraical denominators.

Note. The result obtained by dividing a number p by a number q is clearly p times as great as the number obtained by dividing unity by q. Hence follow the two ways of looking at numerical fractions; $\frac{3}{5}$ may be regarded as the result obtained by dividing 3 by 5 or as three times the result obtained by dividing 1 by 5.

134. The value of a fraction is unaltered by multiplying or dividing both numerator and denominator by the same quantity.

This follows from the axiom that if equals be multiplied by equals, the products are equals.

For suppose $qx = p$, then $x = \frac{p}{q}$.

Multiply both sides by m,

then $mqx = mp$ and $x = \frac{mp}{mq}$.

Hence $\frac{p}{q} = \frac{mp}{mq}$.

Also since $\frac{mp}{mq} = \frac{p}{q}$, it is seen that **both numerator and denominator may be divided by the same quantity without altering the value of the fraction** (provided that quantity is not equal to nothing, § 108).

It follows from this rule that factors common to both numerator and denominator may be divided out just as in Arithmetic; and that any number of fractions may be reduced to a common denominator.

Examples XVI a.

Reduce to their simplest form:

1. $\frac{3abx^2y}{15a^2bx}$.
2. $\frac{5a^2b^3x^4}{2ab^2x^3}$.
3. $\frac{24a^4b^5c^7}{18a^5b^3c^9}$.

4. $\frac{x^2-a^2}{x^2-2ax+a^2}$. 5. $\frac{a^2-2ab-3b^2}{a^2-4ab-5b^2}$. 6. $\frac{x^2-2ax-48a^2}{x^2+ax-42a^2}$.

7. $\frac{x^3+2x^2-x-2}{x^3+3x^2-4x-12}$. 8. $\frac{xy+4ab+2ay+2bx}{xy-4ab+2ay-2bx}$.

9. $\frac{a^2-2ab+b^2+a-b}{a^2-2ab+b^2+a^2-b^2}$.

Express the following fractions as fractions with the same denominator:

10. $\frac{2}{a}, \frac{3}{b}, \frac{x}{ab}$. 11. $\frac{x^2}{3ab}, \frac{2y^2}{5bc}, \frac{3xy}{4ac}$.

12. $\frac{x-a}{x+a}$ and $\frac{x+a}{x-a}$. 13. $\frac{a-2b}{a-b}$ and $\frac{a+2b}{a+b}$.

14. $\frac{2x}{3a}$ and $\frac{2x-3a}{2x+3a}$. 15. $\frac{p}{p+2q}$ and $\frac{q}{p-2q}$.

16. By taking a rectangle 5 by 1 to represent unity, show that three divided by five equals three-fifths.

17. By taking a rectangle 5 by 4 to represent unity, show graphically that $\frac{4}{5} = \frac{16}{20}$.

18. Show graphically that $\frac{3}{4} = \frac{9}{12}$ and that $\frac{8}{12} = \frac{2}{3}$.

19. Show graphically that $\frac{2}{3} + \frac{1}{5} = \frac{11}{15}$.

By reducing to the least common denominator, simplify:

20. $\frac{1}{2a} + \frac{2}{3a} - \frac{3}{5a}$. 21. $\frac{a}{2b} + \frac{2b}{3a}$.

22. $\frac{3a}{4b} + \frac{2b}{3c} - \frac{5c}{12a}$. 23. $\frac{3}{ab} + \frac{2}{bc} - \frac{3}{ac}$.

24. $\frac{3x}{ab} + \frac{2y}{bc} - \frac{3z}{ac}$. 25. $\frac{3ab}{x} + \frac{2bc}{y} - \frac{3ac}{z}$.

26. $\frac{ab}{3x} + \frac{bc}{2y} - \frac{ac}{3z}$. 27. $\frac{2ab}{3x} - \frac{5bc}{2y} + \frac{4ac}{3z}$.

28. $\frac{1}{x+a} + \frac{1}{x-a}$. 29. $\frac{x}{x+a} - \frac{x}{x-a}$.

Just as $\frac{19}{5}$ can be reduced to the form $3\frac{4}{5}$, reduce the following fractions to similar forms:

30. $\frac{2a+3}{a}$. 31. $\frac{3x+4a}{x}$. 32. $\frac{3x^2-4ax+5a^3}{x}$.

33. $\frac{2a+3b}{a+b}$. 34. $\frac{3a-4c}{a-c}$. 35. $\frac{4a-7x}{a+x}$.

36. $\dfrac{a^2-2ab+3b^2}{a-b}$.

37. $\dfrac{a^2+3ab+3b^2}{a+2b}$.

38. $\dfrac{a^2-5ab+6b^2}{(a-6b)}$.

Solve:

39. $\dfrac{x}{a}-\dfrac{1}{b}=\dfrac{x}{b}-\dfrac{1}{a}$.

40. $\dfrac{x}{ab}+\dfrac{3}{bc}=\dfrac{1}{ac}-\dfrac{x}{ab}$.

41. $\dfrac{a}{bx}-\dfrac{b}{ax}=a^2-b^2$.

42. $\dfrac{x}{a}+\dfrac{y}{b}=1-\dfrac{x}{b}$, $\dfrac{y}{a}+\dfrac{x}{b}=1+\dfrac{y}{b}$.

135. Addition and Subtraction of Fractions.

Suppose $x=\dfrac{p}{q}$ and $y=\dfrac{r}{s}$,

then $x=\dfrac{ps}{qs}$ and $y=\dfrac{qr}{qs}$.

$\therefore$ $qsx=ps$,

and $qsy=qr$.

Add $qs(x+y)=ps+qr$.

$\therefore$ $x+y=\dfrac{ps+qr}{qs}$.

The same method can be applied however many fractions there may be, and whether the connecting signs are + or −. Hence follows the rule:

To add or subtract fractions. Reduce all the fractions to the common denominator of lowest dimensions, and then add or subtract the new numerators.

As a rule the common denominator is left in factors and not multiplied out.

As is pointed out in § 78 the line of a fraction has the effect of a vinculum or bracket; so that a minus sign before the fraction changes the sign of all the terms in the numerator. If examples are at first worked as in the following example, there is less chance of making a mistake.

$$\frac{x-a}{b}-\frac{x-b}{a}=\frac{a(x-a)-b(x-b)}{ab}$$
$$=\frac{ax-a^2-bx+b^2}{ab}$$
$$=\frac{(a-b)x-(a^2-b^2)}{ab}.$$

When the student has had a fair amount of practice the first step may be omitted.

$$\frac{2x+3a}{2b}-\frac{3x-2b}{2a}=\frac{2ax+3a^2-3bx+2b^2}{2ab}$$
$$=\frac{(2a-3b)x+(3a^2-2b^2)}{2ab}.$$

Simplify $-\dfrac{x-10}{x^2+x-12}-\dfrac{x+22}{x^2-x-20}+\dfrac{x+1}{x^2-8x+15}$.

Expression:

$$=-\frac{x-10}{(x+4)(x-3)}-\frac{x+22}{(x-5)(x+4)}+\frac{x+1}{(x-3)(x-5)}$$
$$=\frac{-(x-10)(x-5)-(x+22)(x-3)+(x+1)(x+4)}{(x-3)(x+4)(x-5)}$$
$$=\frac{-(x^2-15x+50)-(x^2+19x-66)+(x^2+5x+4)}{(x-3)(x+4)(x-5)}$$
$$=\frac{-x^2+15x-50-x^2-19x+66+x^2+5x+4}{(x-3)(x+4)(x-5)}$$
$$=\frac{-x^2+x+20}{(x-3)(x+4)(x-5)}$$
$$=-\frac{x^2-x-20}{(x-3)(x+4)(x-5)}$$
$$=-\frac{(x-5)(x+4)}{(x-3)(x+4)(x-5)}$$
$$=-\frac{1}{x-3}.$$

The signs of equality should be kept in a straight line one under the other. After a time the student will be able to leave out some of the steps, but he should always recollect that accuracy is more important than speed, and that untidy, badly-arranged work is more likely to be wrong than neat work.

Examples XVI b.

Simplify the following expressions:

1. $\frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}$.

2. $\frac{b^2c^2}{d}+\frac{a^2c^2}{b}-\frac{b^2d^2}{c}$.

3. $-\frac{x+1}{2}-\frac{3x+1}{4}+\frac{5x+1}{6}$.

4. $\frac{2x+3}{5}-\frac{3x-2}{4}+\frac{7x-2}{20}$.

5. $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}$.

6. $\frac{x(1-3x)}{3}-\frac{3(x-2)}{4}-\frac{x^2+3}{6}-\frac{7(x^2-1)}{36}$.

7. $\frac{1-x}{2}-\frac{2(x-2)}{3}+\frac{3(1-2x)}{4}$.

8. $\frac{1}{x+1}+\frac{1}{x+5}$.

9. $\frac{3}{x-1}-\frac{1}{x+3}$.

10. $\frac{2a^2}{a^2-b^2}+\frac{b}{a+b}$.

11. $\frac{2a-b}{a-b}-\frac{2a+b}{a+b}$

12. $\frac{3}{x+2}-\frac{2}{x+3}+\frac{x+1}{(x+2)(x+3)}$.

13. $\frac{x-2}{x-1}-\frac{x-1}{x-2}-\frac{3}{(x-1)(x-2)}$.

14. $\frac{x-1}{2x}-\frac{x^2-2x+1}{4x^2}-\frac{x^3-x-1}{4x^3}$.

15. $\frac{3}{x-3}-\frac{1}{x+5}-\frac{2}{x-7}$.

16. $\frac{y}{x(y-x)}-\frac{x}{y(x+y)}+\frac{2y}{x^2-y^2}$.

17. $\frac{x}{x^2-3x+2}+\frac{2}{x^2-1}-\frac{6}{x^2-x-2}$.

18. $\frac{3}{1+a}-\frac{2}{1-a}-\frac{5a}{a^2-1}$.

19. $\frac{3}{x+1}-\frac{2}{x^2+3x+2}-\frac{2}{x+2}$.

20. $\frac{6a}{a^2-4}-\frac{2}{2+a}+\frac{3}{2-a}$.

136. Multiplication and Division of Fractions.

Suppose $x = \frac{p}{q}$ and $y = \frac{r}{s}$,

then $qx = p$,

$sy = r$.

Multiply together the two left-hand sides and the two right-hand sides,

$$qsxy = pr.$$

Divide by qs, $$xy = \frac{pr}{qs}.$$

Any number of fractions could be dealt with in a similar way.

Hence the product of any number of fractions is a fraction whose numerator is the product of the original numerators and whose denominator is the product of the original denominators. By § 134, any factor common to the numerator and denominator may be divided out.

Again if $x = \frac{p}{q}$ and $y = \frac{r}{s}$,

then $qx = p$

and $sy = r$.

Dividing $\frac{qx}{sy} = \frac{p}{r}$.

Multiply by s $\frac{qx}{y} = \frac{ps}{r}$

Divide by q $\frac{x}{y} = \frac{ps}{qr} = \frac{p}{q} \times \frac{s}{r}$.

Hence we have the rule for division:

To divide by any fraction invert the fraction and multiply.

In all cases involving multiplication and division of fractions, the first step is to resolve all the numerators and denominators into factors, and then cancel factors common to the numerator and denominator.

Example I.

Simplify $\dfrac{a^2+ab}{a^2-ab} \times \dfrac{ab-b^2}{ab+b^2} \times \dfrac{ab-bc}{ab+bc}$.

$$\text{Expression} = \frac{a^2+ab}{a^2-ab} \times \frac{ab-b^2}{ab+b^2} \times \frac{ab-bc}{ab+bc}$$

$$= \frac{\cancel{a}(a+b)}{\cancel{a}(a-b)} \times \frac{\cancel{b}(a-b)}{\cancel{b}(a+b)} \times \frac{\cancel{b}(a-c)}{\cancel{b}(a+c)}$$

$$= \frac{a-c}{a+c}.$$

Examples XVIe.

Simplify the following expressions :

1. $\dfrac{2ab^2c}{3a^2bc^2} \times \dfrac{4a^2bc^3}{5ab^2c^4} \times \dfrac{3abc}{2a^2bc^3}$.
2. $\dfrac{3ab}{2cx} \times \dfrac{5by}{4ax} \times \dfrac{3cz}{2bx}$.
3. $\dfrac{a(x-b)}{bx+ab} \times \dfrac{b^2x+ab^2}{a^2x+a^3} \times \dfrac{x^2-a^2}{x^2-b^2}$.
4. $\dfrac{3ab^2}{4bc^2} \times \dfrac{2b^2c}{7ac} \div \dfrac{3a^2b^2}{4bc}$.
5. $\dfrac{3ab}{2cx} \times \dfrac{5by}{4ax} \div \dfrac{3cz}{2bx}$.
6. $\dfrac{x^2-5x+6}{x^2-16} \times \dfrac{x^2+7x+12}{x^2-8x+15} \times \dfrac{x^2+x-20}{x^2+x-6}$.
7. $\dfrac{4x^2-9}{2x^2+5x+3} \div \dfrac{2x-3}{x(x+1)}$.
8. $\dfrac{6x^2-ax-2a^2}{ax-a^2} \times \dfrac{x-a}{9x^2-4a^2} \div \dfrac{a+2x}{a(2a+3x)}$
9. $\dfrac{x^2-(y-z)^2}{y^2-(z-x)^2} \div \dfrac{(y+z)^2-x^2}{z^2-(x-y)^2}$.
10. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \times \dfrac{a-b}{a^2+ab}$.
11. $\dfrac{2x^2+x-1}{x^2-4x+3} \times \dfrac{2x^2-5x+3}{6x^2+x-2} \times \dfrac{3x^2-7x-6}{2x^2-7x+6}$
12. $\dfrac{(x-y)^2-z^2}{xy-y^2-yz} \times \dfrac{z}{x^2+xy-xz} \div \dfrac{xz-yz+z^2}{x^2-(y-z)^2}$.

13. $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\div\left(\frac{1}{a}-\frac{1}{b}\right)$.

14. $\left(\frac{1}{2a}+\frac{1}{3b}\right)\times 6ab$.

15. $\frac{x+2a}{2x-3a}\left(\frac{1}{x+2a}+\frac{1}{x-5a}\right)$.

16. $\left(\frac{1}{a}+\frac{1}{ab^3}\right)\div\left(b-1+\frac{1}{b}\right)$.

17. $\left(\frac{a}{a-x}-\frac{x}{a+x}+\frac{2ax}{a^2-x^2}\right)\div(a+x)^2$.

18. What must $\frac{x-3}{x-4}$ be multiplied by to give $\frac{x^2-5x+6}{x^2-7x+12}$ as product?

19. Find the quotient when $\frac{x+4}{x-1}$ is divided by $\frac{3x^2+10x-8}{4x^2+x-5}$.

20. Solve $\frac{(a-b)x}{a+b}=a^2-b^2$.

21. Find the missing term in the following identity:

$$\frac{x^2-5x+6}{x^2-3x-4}\times\frac{x^2-x+\quad}{x^2+2x-15}\div\frac{x^2+5x-14}{x^2+6x+5}=\frac{x^2-9}{x^2+4x-21}.$$

22. What must $\frac{x-2}{x-3}-\frac{x-4}{x-5}$ be multiplied by in order that the product shall contain no fractions? What is the simplest form of the product?

23. Solve the equations:

(i) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=\frac{x}{(x-3)(x-5)}$;

(ii) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=\frac{x^2-8x+13}{(x-3)(x-5)}$;

(iii) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=-\frac{2}{(x-3)(x-5)}$;

(iv) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=0$.

24. Simplify $\left(1+\frac{b}{a}\right)\left(1+\frac{c}{a}\right)\div\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)$.

25. Simplify $\left(1+\frac{2a}{1+a}\right)\left(1+\frac{2a}{1-a}\right)$.

137. Special points in connexion with fractions.

(i) Reduce all fractions to their lowest terms before adding or subtracting.

(ii) Factorize all numerators or denominators before multiplying or dividing.

(iii) $\frac{a}{b} \times b = a$, $\frac{a}{b} \times bc = ac$. That is:

Any fraction can be reduced to an integral expression by multiplying it by its denominator or by a multiple of its denominator.

Hence the algebraic sum of any number of fractions can be reduced to an integral form by multiplying by the L. C. M. of all the denominators.

This rule is continually being used when equations that contain fractions are to be solved.

(iv) The value of a fraction is unaltered when the numerator and denominator are both multiplied by the same quantity.

This has already been given as the first rule in dealing with fractions. It is here mentioned again as number (iv) because, together with number (iii) above, it gives the easiest method of simplifying complicated fractions.

138. Miscellaneous worked examples in Fractions.

Example I.

Simplify $\dfrac{x^2-2ax+a^2}{x^2-a^2} + \dfrac{3ax-15a^2}{x^2-4ax-5a^2} - \dfrac{3x^2+4ax-4a^2}{x^2+3ax+2a^2}$.

Expression

$$= \frac{(x-a)^2}{(x-a)(x+a)} + \frac{3a(x-5a)}{(x+a)(x-5a)} - \frac{(3x-2a)(x+2a)}{(x+a)(x+2a)}$$

$$= \frac{x-a}{x+a} + \frac{3a}{x+a} - \frac{3x-2a}{x+a}$$

$$= \frac{-2x+4a}{x+a}$$

$$= -\frac{2(x-2a)}{x+a}.$$

Example II. *Simplify* $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$.

$$\text{Fraction} = \frac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$$

$$= \frac{a^2-b^2}{a^2+b^2}.$$

(Multiplying num. and den. by ab, § 137 iv, in order to clear them of fractions, § 137 iii.)

Example III. *Simplify* $\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{x}{x^2-a^2}$.

[In all examples it is advisable to keep the letters in the same order; here a comes before x in two fractions but x before a in the third fraction. The first step is to get the same order throughout. Recollect that $a+b=b+a$, but $a-b=-(b-a)$.]

$$\text{Expression}=\frac{1}{a+x}+\frac{1}{a-x}+\frac{-x}{a^2-x^2}$$ (Multiplying num. and den. of third fraction by -1.)

$$=\frac{a-x+a+x-x}{a^2-x^2}$$

$$=\frac{2a-x}{a^2-x^2}.$$

If we multiply both numerator and denominator by -1, we see that the answer may be written $\dfrac{x-2a}{x^2-a^2}$.

Example IV.

Simplify $\left(1-\dfrac{2ab}{a^2+b^2}\right)\times\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\div\left(a-\dfrac{2ab}{a+b}\right)$.

[The quantities in brackets should be brought to a common denominator before the brackets are removed. Such expressions as this should be kept as single expressions throughout the work, and the parts should not be simplified by separate sums.]

Wrong way.

$$1-\frac{2ab}{a^2+b^2}=\frac{a^2+b^2-2ab}{a^2+b^2},\qquad a-\frac{2ab}{a+b}=\frac{a^2+ab-2ab}{a+b}$$

$$=\frac{a^2-ab}{a+b},$$

$$\frac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}=\frac{a^2+b^2}{a^2-b^2}.$$

$$\text{Expression}=\frac{a^2-2ab+b^2}{a^2+b^2}\times\frac{a^2+b^2}{a^2-b^2}\times\frac{a+b}{a^2-ab},$$

&c.

Right way.

$$\text{Expression} = \left(1 - \frac{2ab}{a^2+b^2}\right) \times \frac{\frac{a}{b} + \frac{b}{a}}{\frac{a}{b} - \frac{b}{a}} \div \left(a - \frac{2ab}{a+b}\right)$$

$$= \frac{a^2+b^2-2ab}{a^2+b^2} \times \frac{a^2+b^2}{a^2-b^2} \div \frac{a^2+ab-2ab}{a+b}$$

$$= \frac{(a-b)^2}{(a-b)(a+b)} \times \frac{a+b}{a(a-b)}$$

$$= \frac{1}{a}.$$

In the first line each expression in brackets is reduced to its common denominator, and the second fraction is simplified by multiplying throughout by ab.

In the second line the common factor a^2+b^2 is omitted, and the numerators and denominators are resolved into factors.

Example V. When several fractions are to be added or subtracted, much labour can be saved by a little thought as to the arrangement of the fractions. Side multiplications can often be avoided by taking only two or three fractions at a time, so that there is no necessity to multiply together more than two binomial expressions in any one step.

Simplify $\dfrac{1}{6x-18} - \dfrac{1}{6x+18} - \dfrac{1}{x^2+9} + \dfrac{18}{x^4+81}$.

$$\text{Expression} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)} - \frac{1}{x^2+9} + \frac{18}{x^4+81}$$

$$= \frac{x+3-x+3}{6(x^2-9)} - \frac{1}{x^2+9} + \frac{18}{x^4+81}$$

$$= \frac{x^2+9-x^2+9}{x^4-81} + \frac{18}{x^4+81}$$

$$= 18\left\{\frac{1}{x^4-81} + \frac{1}{x^4+81}\right\}$$

$$= \frac{36x^4}{x^8-6561}.$$

139. Equations containing Fractions.

The following examples show how to deal with equations containing algebraical fractions:

Example I.

Solve $\dfrac{3}{x-2}+\dfrac{4}{x-3}=\dfrac{7}{x-4}$.

Multiply by L. C. M. of denominators. [§ 137 (iii).]

$$3(x-3)(x-4)+4(x-2)(x-4)=7(x-2)(x-3).$$

Remove brackets

$$3x^2-21x+36+4x^2-24x+32=7x^2-35x+42$$

Transpose and collect $-10x=-26$

$$x=2\tfrac{3}{5}.$$

Example II.

Solve $\dfrac{2x-2}{x-2}-\dfrac{x+4}{x+3}=\dfrac{3x-8}{3x-11}$.

If we were to multiply by the L. C. M. of the denominators we should have to multiply out $(2x-2)(x+3)(3x-11)$ and two similar expressions. This would involve three sums at the side, as these multiplications are too long for mental work. It is always well to avoid side work, so the following method may be adopted.

Bring each side to its own L. C. D.

$$\frac{2x^2+4x-6-x^2-2x+8}{(x-2)(x+3)}=\frac{3x-8}{3x-11}.$$

Collect the numerators

$$\frac{x^2+2x+2}{x^2+x-6}=\frac{3x-8}{3x-11}.$$

Mentally resolve into factors and see if any factor divides out. Here there is no common factor.

Multiply now by the L. C. M. of the denominators

$$(x^2+2x+2)(3x-11)=(x^2+x-6)(3x-8).$$

Remove brackets

$$\begin{aligned} 3x^3+6x^2+6x &= 3x^3+3x^2-18x \\ -11x^2-22x-22 &\quad -8x^2-8x+48. \end{aligned}$$

Collect $-5x^2-16x-22 = -5x^2-26x+48.$

Transpose and collect $10x=70.$

Divide by 10 $x=7.$

Verify L. H. S. $=\frac{12}{5}-\frac{11}{10}=\frac{13}{10}.$

R. H. S $=\frac{13}{10}.$

Example III.

Solve $\dfrac{2x-2}{x-2}-\dfrac{x+4}{x+3}=\dfrac{3x-8}{3x-11}$.

[When the numerators are not of lower dimensions than the denominators, equations are often simplified by dividing each numerator by its denominator until the remainder is of lower dimensions than the denominator. This division can usually be worked mentally just as $\frac{37}{8}=4\frac{5}{8}$ is done mentally, e. g. $\dfrac{4x-16}{x-2}=4-\dfrac{8}{x-2}$.]

$$\frac{2x-2}{x-2}-\frac{x+4}{x+3}=\frac{3x-8}{3x-11}.$$

Divide numerators by denominators

$$2+\frac{2}{x-2}-\left(1+\frac{1}{x+3}\right)=1+\frac{3}{3x-11}.$$

Simplify: $$\frac{2}{x-2}-\frac{1}{x+3}=\frac{3}{3x-11}.$$

Multiply by L. C. D.:

$$2(3x^2-2x-33)-(3x^2-17x+22)=3(x^2+x-6)$$

Remove brackets:

$$6x^2-4x-66-3x^2+17x-22=3x^2+3x-18.$$

Transpose and collect:

$$10x=70$$
$$x=7.$$

This agrees with Example II, but the work has been much less. This method is always shorter when the quotients obtained by division cancel.

Example IV.

Solve $\dfrac{10}{x-4}-\dfrac{5}{x-3}=\dfrac{4}{x-6}+\dfrac{1}{x-1}$.

To avoid side multiplication reduce each side to its L. C. D.

$$\frac{10x-30-5x+20}{(x-4)(x-3)}=\frac{4x-4+x-6}{(x-6)(x-1)}.$$

Collecting the numerators and multiplying out the denominators

$$\frac{5x-10}{x^2-7x+12}=\frac{5x-10}{x^2-7x+6}.$$

Dividing by $5x-10$ and multiplying across

$$x^2-7x+6=x^2-7x+12.$$

Transpose and collect: $0=6$,

which is impossible, and apparently the equation has no solution.

There is, however, an error in the working, which has already been discussed in Chap. XIII, § 108.

Examples XVI d.

Simplify:

1. $\frac{x}{y}\left(\frac{1}{x-y}+\frac{2}{x+y}-\frac{3}{x}\right)$.

2. $\frac{ab-b^2}{ab+a^2}-\frac{ab-a^2}{ab+b^2}$.

3. $\frac{1}{2-x}+\frac{1}{3-x}+\frac{2}{x-6}$.

4. $\frac{a^3-b^3}{a^2+b^2}-\frac{a^2+b^2}{a-b}$.

5. $\frac{1}{6(1+x)}+\frac{1}{6(1-x)}+\frac{1}{3(1+x^2)}$.

6. $\frac{x+y}{x^3-y^3}+\frac{x-y}{x^3+y^3}-\frac{2(x^2-y^2)}{x^4+x^2y^2+y^4}$.

7. $\frac{1}{ax-2bx}-\frac{4b}{a^2x-4b^2x}+\frac{1}{ax+2bx}$.

8. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

9. $\left(\frac{a}{a+x}+\frac{x}{a-x}\right)\div\left(\frac{a}{a-x}-\frac{x}{a+x}\right)$.

10. $\frac{p+2}{2}-\frac{p}{p+2}-\frac{p^3-2p^2}{2p^2-8}$.

11. $\left\{\frac{p}{(p-1}-\frac{p}{p+1}\right\}\div\frac{1}{8}\left\{\frac{1}{p-1}-\frac{1}{p+1}+\frac{1}{1-p^2}\right\}$.

12. $\frac{x^3+6x^2-2x-5}{x^3+9x^2+19x+10}$.

13. $\frac{a-b}{a(a+b)}-\frac{a+b}{b(b-a)}-\frac{3a-b}{a^2-b^2}$.

14. $\left(\frac{x-a}{x+a}+\frac{x+a}{x-a}\right)\div\left(\frac{a-x}{a+x}-\frac{a+x}{a-x}\right)$.

15. $\frac{2x+3}{x-x^2}+\frac{5-x}{x-1}-\frac{3}{x}$.

16. $\left(x+1+\frac{8}{x-5}\right)\left(x-1-\frac{8}{x-5}\right)$.

17. $\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}-\frac{a+b}{(c-a)(c-b)}$.

18. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2-ca}{(b-c)(b-a)}+\dfrac{c^2-ab}{(c-a)(c-b)}$.

19. $\dfrac{a}{(a-x)^2}-\dfrac{3a}{2a^2-ax-x^2}+\dfrac{1}{2a+x}$.

20. $\dfrac{a-\dfrac{b^2}{a}}{\dfrac{a^3}{b^2}-\dfrac{b^3}{a}}\times\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}\right)$.

21. $\left(\dfrac{3}{2a-b}-\dfrac{2}{3a+b}\right)\left(1+\dfrac{2a}{a+b}\right)$.

22. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

23. $\dfrac{x-1}{x-2}\left\{1-\dfrac{1}{x+1}-\dfrac{2}{x^2-1}\right\}$.

24. $\left\{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right\}\div\left\{\dfrac{a^2+ax}{ax-x^2}-\dfrac{a^2-ax}{ax+x^2}\right\}$.

25. $\left\{\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right\}\times\left\{\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right\}$.

Solve the following equations:

26. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{2}{x}$.

27. $\dfrac{1}{x}+\dfrac{3}{x-1}=\dfrac{4}{x-2}$.

28. $\dfrac{x-a}{1+b}-\dfrac{x-b}{1+a}=b-a$.

29. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$

30. $\dfrac{x+a+c}{x+b+c}=\dfrac{b}{a}$.

31. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$.

32. $\dfrac{2(3-4x)}{3-x}+\dfrac{3}{1-x}=8$.

33. $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$.

34. $\dfrac{x}{2x-1}-\dfrac{\frac{1}{2}x-\frac{1}{4}}{4x^2-1}+\dfrac{1}{2x+1}=\frac{1}{2}$.

Simplify:

35. $\dfrac{\dfrac{a+2b}{a+b}+\dfrac{a}{b}}{\dfrac{a+2b}{b}-\dfrac{a}{a+b}}\div\dfrac{\dfrac{a^2}{b^2}+\dfrac{1}{a}}{\dfrac{a}{b^2}-\dfrac{1}{b}+\dfrac{1}{a}}$.

36. $\dfrac{(a-b)^2-c^2+(a+c)^2-b^2+(c-b)^2-a^2}{(b+c)^2-a^2+(c+a)^2-b^2+(a+b)^2-c^2} \times \dfrac{a+b+c}{a-b+c}$.

37. $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2} - \dfrac{x^3-\dfrac{1}{x^3}}{x^2+\dfrac{1}{x^2}+1}$.

38. $\dfrac{(1-x^2)(1-y^2)-4xy}{(1-x)(1-y)-2xy}$.

39. $\dfrac{x^2+xy+y^2-a(x+y)+a^2}{x+y+a} + \dfrac{x^2-xy+y^2+a(x-y)-a^2}{x-y+a}$.

40. $\dfrac{\dfrac{x^2}{x^2+y^2}}{\dfrac{x+y}{}-y} + \dfrac{\dfrac{y^2}{x^2+y^2}}{\dfrac{x+y}{}-x}$.

41. $\dfrac{(x+a)^3+(x+b)^3-(2x+a+b)^3}{(x+a)(x+b)(2x+a+b)}$.

42. $\dfrac{2y}{x+y} + \dfrac{1+\dfrac{x}{y}+\dfrac{x^2}{y^2}}{1-\dfrac{x}{y}+\dfrac{x^2}{y^2}} \times \dfrac{y^3+x^3}{y^3-x^3}$.

43. $\dfrac{1}{x-1} \div \left(\dfrac{x+\frac{1}{2}}{x-1} - \dfrac{x+1}{x-\frac{1}{2}}\right)$.

44. $\dfrac{x+y}{x+2y+\dfrac{3y^2}{x-2y}} \times \dfrac{x-y}{x^2-xy-2y^2}$.

45. $\dfrac{1}{(1-a)(1-b)} - \dfrac{a^2}{(1-a)(a-b)} + \dfrac{b^2}{(1-b)(a-b)}$.

Solve the following equations:

46. $\dfrac{3}{x-3} + \dfrac{4}{x-4} = \dfrac{25}{x(x-7)+12}$.

47. $\dfrac{1}{6x+9} + \dfrac{2}{10x+1} = \dfrac{2}{2x+3}$.

48. $\dfrac{x-2a}{b+c-a} + \dfrac{x-2b}{c+a-b} + \dfrac{x}{a+b+c} = 3$.

49. $\dfrac{1}{(x+a)} - \dfrac{1}{x+b} - \dfrac{1}{x+a+b} + \dfrac{1}{x+2b} = 0$.

50. $\dfrac{7}{x+2} - \dfrac{10}{x-5} + \dfrac{3}{x+4} = 0$.

Examples XVI.

1. Simplify $\dfrac{a^2-2ab+b^2}{a^2+3ab-4b^2}$.

2. Simplify $\dfrac{7x+6}{2(x+2)}+\dfrac{4}{x+2}-\dfrac{7}{2}$.

3. Simplify $\dfrac{x^2-4}{x^2+3x+2}\times\dfrac{(x+1)^2}{x^2-x-12}\div\dfrac{x^2-x-2}{x^2-4x}$.

4. Solve $\dfrac{5x-(x-3)}{3x+(x-2)}=\dfrac{9}{4}$.

5. A had £50 more than B; A paid a third part of his money to B, and B paid back a fifth part of what he then had: the result was that B had £30 more than A. How much had each at first?

---

6. Simplify $\dfrac{a-b}{b}+\dfrac{2a}{a-b}-\dfrac{a^2}{b(a-b)}$.

7. Reduce $\dfrac{2x^2-x-1}{3x^2-x-2}$ to its lowest terms.

8. Simplify $\dfrac{2}{a+3b}-\dfrac{1}{a+2b}$, and divide the result by $a+b$.

9. Solve $\dfrac{1}{x}+\dfrac{2}{x+1}=\dfrac{3}{x+2}$.

10. A man buys oranges at the rate of p a shilling and by selling them at q pence a dozen makes a profit of r per cent. Show that $25pq-36r=3600$.

---

11. Simplify $\dfrac{2}{a-2b}+\dfrac{1}{a+2b}-\dfrac{4a}{a^2-4b^2}$.

12. Simplify $\dfrac{a^2-ab}{2bc-ab}\times\dfrac{b^2+bc}{ac-bc}\times\dfrac{2ac-bc}{ab+ac}$.

13. Reduce $\dfrac{(x-y)^2}{(x^2-y^2)(x^3-y^3)}$ to its lowest terms.

14. Solve $\dfrac{x-5a}{a+b}+\dfrac{x+b}{a-b}+1=0$.

15. If p men do the same amount of work as q boys, and a men earn £10 in 6 days, how much will b boys earn in 10 days?

---

16. Simplify $\frac{1}{2a+b}+\frac{1}{2a-b}-\frac{3a}{4a^2-b^2}$.

17. Solve $\frac{2x-15}{2x-6}-\frac{1}{2(x+7)}=\frac{x-7}{x+7}$, and verify the answer.

18. Simplify $\frac{3abc-3b^2c}{a-bc-b(1-c)}$.

19. Show that $\frac{1}{x-a}+\frac{1}{x-b}$ has the same value when

$$x=a+b, \text{ as when } x=\frac{2ab}{a+b}.$$

20. A can do a piece of work in x hours, B can do it in y hours. How long will A and B take to do it working together? After they have worked together for z hours, what fractional part remains to be done?

---

21. Simplify $\left(\frac{1}{x+3}-\frac{1}{2x+3}\right)\left(2+\frac{3}{x}\right)$.

22. Simplify $\frac{\frac{1}{a}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{b}\left(\frac{-1}{a}+\frac{1}{b}\right)}{\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}+\frac{1}{b}\right)^2}$.

23. Solve $\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+4}$.

24. Simplify $\frac{x^3-y^3}{(x+y)^2}\times\frac{x^3+y^3}{x^2+xy-2y^2}\div\frac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}$.

25. A man bought a certain number of sheep for forty guineas. He lost one-tenth of them, and sold the rest for half a guinea a head more than he had given for them and thus gained five guineas. How many sheep did he buy?

---

26. Simplify $-7ac-\{2c(a-3b)-3a(5c-2b)\}$; and find its value when $c=\frac{ab}{a+b}$.

27. Simplify $\frac{a^2-b^2}{a^2b^2}+\frac{b^2-c^2}{b^2c^2}+\frac{c^2-a^2}{c^2a^2}$.

28. Simplify $\frac{x^2-x-6}{x^2-3x+2}\times\frac{x^2+x-6}{x^2+x-2}\times\frac{x^2-1}{x^2+6x+9}$.

29. Solve $\frac{1}{x-2} - \frac{1}{3-x} - 1\frac{1}{2} = 0$.

30. A party consists of men, women, and boys; the men are one more than one-sixth of the whole, the women two more than one-quarter of the whole, and the boys five more than one-half of the whole. How many are there altogether?

---

31. Reduce to its lowest terms $\frac{x^6+3x^4y+x^2y^4+3y^5}{x^6+3x^4y+2x^2y^4+6y^5}$.

32. Find the value of $\frac{a-p}{b-p}$ when $p = \frac{ab}{a+b}$.

33. Simplify $\frac{p}{p^2+5p+6} + \frac{15}{p^2+9p+14} - \frac{12}{p^2+10p+21}$.

34. Solve $\frac{6x+8}{2x+1} - \frac{2x+38}{x+12} = 1$.

35. Some smugglers discovered a cave which would exactly hold the cargo of their boat, viz. 13 bales of cotton and 33 casks of rum. Whilst they were unloading, a custom house cutter came in sight, so they sailed away with 9 casks and 5 bales, having filled only $\frac{2}{3}$ of the cave. How many bales would the cave hold?

---

36. If $P = q-r$, $Q = r-p$, $R = p-q$, find the value of

$$\frac{1}{QR} + \frac{1}{RP} + \frac{2}{PQ}.$$

37. Simplify $\frac{x^6+8}{x^6-8} \div \frac{x^2+2}{x^2-2}$.

38. Simplify $\frac{p}{p-q} + \frac{q}{p+q} + \frac{pq}{p^2-q^2} - \frac{q^2}{p^2+q^2}$.

39. Solve $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$.

40. A shepherd in time of war was robbed by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep. A third party took $\frac{1}{2}$ of what now remained and $\frac{1}{2}$ of a sheep. He had 25 sheep left; how many had he at first? Verify.

---

41. Simplify $\frac{p}{q-r} + \frac{q}{r-p} + \frac{pq-q^2}{(q-r)(r-p)}$.

42. Simplify $\dfrac{\dfrac{a}{1+a}+\dfrac{a}{1-a}}{\dfrac{1}{1-a}-\dfrac{1}{1+a}}$.

43. Find the value of $6\sqrt{ax^2}-\dfrac{2ax^3}{(a-x)^2}-a^2$ when $a=4$, $x=2$.

44. Solve $\dfrac{5x+6}{2x-4}=\dfrac{10x-7}{4x-2}$.

45. Find a fraction, such that if 4 be added to its numerator its value is $\frac{1}{2}$ and if 7 be added to the denominator its value is $\frac{1}{3}$.

---

46. Simplify $\dfrac{-2x+5}{4x^2+4x-3}-\dfrac{2x+7}{4x^2+8x+3}+\dfrac{2x-5}{4x^2-1}$.

47. Multiply $\dfrac{x}{y}-1+\dfrac{y}{x}$ by $\dfrac{x}{y}+1-\dfrac{y}{x}$.

48. Simplify $\dfrac{(x^2y^3)^4\left(1+\dfrac{x}{y}\right)^3}{\left(1+\dfrac{y}{x}\right)^4}\times\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^3$.

49. Solve $\dfrac{9}{x+6}+\dfrac{15}{2x+6}=\dfrac{9}{2x+12}+\dfrac{90}{8x+24}$.

50. A does $\frac{3}{5}$ of a piece of work by himself in 30 days, and with the help of B finishes it in 10 days more. How long would the work have taken if they had worked together all the time?

---

51. What must be added to $\dfrac{4x}{(x-1)^2(x+1)}-\dfrac{x+1}{(x-1)^2}+1$ to make 3?

52. Simplify $\dfrac{1-\dfrac{2xy}{x^2+y^2}}{\dfrac{x^3-y^3}{x-y}-3xy}$.

53. When $a=1$, $b=-2$, $c=4$, $d=-3$, find the value of

$$\frac{abcd}{b^2c^2}\times\sqrt{d^3+(c-b)^2}+\frac{2cd}{3ab^2}.$$

54. Find the value of $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}+\dfrac{1}{ax-ac}$ when

$$x=\frac{b}{a}(a-b+c).$$

55. A train starts on a journey of 250 miles at the rate of x miles an hour; after going 100 miles the rate is increased by 5 miles an hour; 2 hours afterwards the rate is again increased by 5 miles an hour. Find an expression for the whole time taken.

---

56. Write two common multiples of the 4th degree in x and y of the following: $2x^2-xy-y^2$, $2x^2-3xy+y^2$. Find the L. C. M.

57. Prove that $\dfrac{y-z}{1+yz}+\dfrac{z-x}{1+zx}+\dfrac{x-y}{1+xy}=\dfrac{(y-z)\,(z-x)\,(x-y)}{(1+yz)(1+zx)(1+xy)}$

58. A man has x sovereigns and y pence. He dies and leaves to each of his m sons one-tenth, and to each of his n daughters one-twelfth of his possessions, while his wife gets the remainder. How much does she get?

59. Simplify $\dfrac{x^2-4xy-21y^2}{x^2-9xy+20y^2}\div\dfrac{x^2-13xy+42y^2}{x^2+3xy-40y^2}$.

60. Solve (i) $\dfrac{x^3-27}{x-3}-\dfrac{x^3+27}{x+3}=6$;

(ii) $\dfrac{1}{x+2+\dfrac{1}{x+3}}=\dfrac{1}{x-2+\dfrac{1}{x-3}}$.

CHAPTER XVII

QUADRATIC EQUATIONS. GRAPHICAL SOLUTIONS

ELEMENTARY THEORY OF QUADRATICS

140. Definition. A quadratic equation is one in which the term of highest dimensions is of the second degree.

In dealing with quadratic equations, fractions, brackets, &c., should be removed in the ordinary way, and all the terms transposed to the left-hand side and collected. All quadratic equations are thus brought to the standard form, viz.

$$ax^2+bx+c=0$$

where a, b, c may be any quantities whatever not containing x.

There are several methods of solution, and it is only by continual practice that the student can tell the best method to adopt in particular cases.

141. Method I. When there are no terms of the first degree in the equation.

Example I. *Solve* $(x+2)^2 = 4x+5$.

$$(x+2)^2 = 4x+5.$$

Remove brackets $x^2+4x+4 = 4x+5$.

Transpose $x^2 = 1$.

Take square root $x = \pm 1$.

It is important to notice that whenever a square root is taken the result may be either positive or negative

[Verify. If $x = +1$, $(x+2)^2 = 9$,
$4x+5 = 9$.
If $x = -1$, $(x+2)^2 = 1$.
$4x+5 = 1$.]

Example II. *Solve* $(x+2)^2 = 4x+6$.

Just as in Example I we get $x^2 = 2$.

Take square root $x = \pm \sqrt{2}$.

By Arithmetic the square root of 2 may be found to any required number of places, and hence the value of x may be found to any required degree of accuracy.

Example III. *Solve* $(x+2)^2 = 4x+2$.

As in Example I we get $x^2 = -2$.

Take square root $x = \pm \sqrt{-2}$.

But no real number when multiplied by itself is negative, therefore $\sqrt{-2}$ does not exist, and is said to be imaginary.

142. Method II. By factors. This is the easiest way of solving quadratics provided the factors are rational (see § 129). By using the method of § 122 quadratics can always be solved by factors, even though the roots are not rational or not real.

Example I. *Solve* $x^2-17x+60 = 0$.

$$x^2-17x+60 = 0.$$

Factorize $(x-5)(x-12) = 0$.

But if the product of two quantities is nothing, one of the quantities must be nothing. Hence

either $x-5 = 0$ or $x-12 = 0$.

$\therefore \quad x = 5$ or 12.

Example II. *Solve* $x^2-8x-13=0$.

$$x^2-8x-13=0.$$

Factorize $x^2-8x+16-29=0,$

$$(x-4)^2-29=0.$$

$$(x-4-\sqrt{29})(x-4+\sqrt{29})=0.$$

$\therefore$ either $x-4-\sqrt{29}=0$ or $x-4+\sqrt{29}=0$;

i.e. $x=4+\sqrt{29}$ or $4-\sqrt{29}$.

143. Method III. By completing the square. We know that $x^2+2ax+a^2=(x+a)^2$, therefore by adding a^2 to x^2+2ax we make an exact square. This process is called **completing the square**; it has already been used in § 122 and § 142 and is of frequent occurrence.

Example I. *Solve* $x^2-8x-13=0$.

$$x^2-8x-13=0.$$

Transpose the constant to the right-hand side,

$$x^2-8x=13.$$

Add the square of half the coefficient of x to each side,

$$x^2-8x+(\tfrac{8}{2})^2=13+16.$$

Take the square root $x-4=\pm\sqrt{29}.$

Transpose $x=4\pm\sqrt{29}.$

If a numerical answer is required the square root may be extracted to any required number of places.

144. Method IV. By formula.

Apply Method III to the standard form,

$$ax^2+bx+c=0.$$

Divide by a and transpose the constant,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

$$\text{i.e. } \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Take square root $x+\dfrac{b}{2a}=\pm\dfrac{\sqrt{b^2-4ac}}{2a}.$

Transpose $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

By giving the correct values to a, b, c, this formula gives the solution of any quadratic.

Example I. *Solve* $3x^2-5x-7=0$.

Here $a=3$, $b=-5$, $c=-7$;

$$\therefore\ x = \frac{5 \pm \sqrt{25+84}}{6} = \frac{5 \pm \sqrt{109}}{6}.$$

Examples XVII a.

Solve, by as many methods as possible, each of the following equations, and in each case find the sum and product of the roots. When the roots are irrational, they should be worked correct to two decimal places.

1. $(x-3)^2=5(5-x)-x$.
2. $(x-3)^2=(x-2)^4$.
3. $(2x-3)^2=(x-2)^2$.
4. $x^2-15x+56=0$.
5. $x^2+49x-660=0$.
6. $3x^2-6x+3=0$.
7. $4x^2-12x+9=0$.
8. $3x^2+7x-6=0$.
9. $x(x-2)-3(x-2)=0$.
10. $x^2(x-2)-3(x-2)=0$.
11. $x^2+x-380=0$.
12. $12x^2-7x-10=0$.
13. $3x^2+4x-5=0$.
14. $3(x^2-2)=4(x-1\frac{1}{2})$.
15. $(x-5)(x-7)=3$.
16. $(x-5)(x-7)=3(x-5)$.
17. $3x^2-11x+8=0$.
18. $y^2-8y-105=0$.
19. $x^2-2ax+a^2=b^2$.
20. $x^2-7ax+12a^2=0$.
21. $3x^2+4x+5=0$.
22. $x^2-3x+1=0$.
23. $x^2-x-600=0$.
24. $2x^2+7x+6=0$.
25. $4x^2+9x+3=0$.
26. $x^2+6x-7=0$.
27. $5x^2+x-8=0$.
28. $x^2+4x-21=0$.
29. $7y^2-8y+3=0$.
30. $x^2-1{\cdot}2x+{\cdot}11=0$.

Without actually solving, write down the sum and product of the roots of the following equations:

31. $x^2-5x+3=0$.
32. $x^2+17x-301=0$.
33. $x^2+91x-1002=0$.
34. $x^2+4bx-18c=0$.
35. $3x^2-44x+77=0$.
36. $3x-4x^2=19$.

Find the simplest equations which have the following roots:

37. $x=5$ or 6.
38. $x=3$ or -2.
39. $x=-1$, or 5, or 6.
40. $x=a+b$ or $a-b$.
41. $x=17$ or $-\frac{2}{3}$.
42. $x=3\frac{1}{2}$ or $-\frac{1}{2}$.

43. Find an expression which vanishes when $x = 3$ or -4.

44. Find an expression which is equal to zero when $y = 7$ or -2.

45. Find a rational expression which equals nothing when $x = \sqrt{2}$. For what other value is this expression equal to nothing?

46. By using the Factor Theorem, show that a quadratic equation cannot have more than two roots.

47. Between what values must x lie in order that $(x-3)(x-4)$ may be negative?

48. Show that if x is a real quantity x^2-4x+6 cannot be negative.

49. Between what values must k lie in order that k^2+k-2 may be positive?

50. Find the values of α and β if $\alpha+\beta = 9$, $\alpha\beta = 18$.

Graphical Methods of Solution.

145. *Solve* $x^2-x-6 = 0$.

There are three different ways of stating the question we are trying to answer, viz.

What value of x will make $x^2-x-6 = 0$?

What value of x will make $x^2-x = 6$?

What value of x will make $x^2 = x+6$?

There are three distinct graphical methods of solving the equation corresponding to these three questions.

Graphical Method I.

Solve $x^2-x-6 = 0$.

What value of x makes $x^2-x-6 = 0$?

Draw the graph of $y = x^2-x-6$ (§ 47), and find the abscissae of the points where it cuts the axis of x.

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
x^2-x	12	6	2	0	0	2	6	12
y	6	0	-4	-6	-6	-4	0	6

In Fig. 27, it is seen that the graph cuts the axis of x at the points A and B; therefore at the points A and B, $y = 0$. But for the point A, $x = -2$, and for the point B, $x = 3$. Hence, when $x = -2$ or 3, $y = 0$; but $y = x^2 - x - 6$.

$\therefore$ the required roots are $x = -2$ or 3.

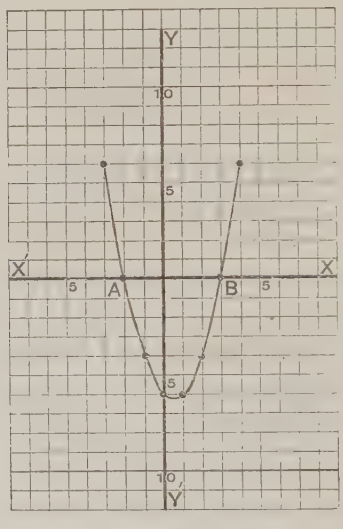


Fig. 27.

The same method can be applied in all cases whether the roots are exact integers or not. With the scale of Fig. 27 the roots can be determined to one decimal place; and by choosing a larger scale to two decimal places.

146. Graphical Method II.

Solve $x^2 - x - 6 = 0$.

What value of x makes $x^2 - x = 6$?

Draw the graph of $y = x^2 - x$ and find the abscissae of the points where it cuts the line $y = +6$.

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
y	12	6	2	0	0	2	6	12

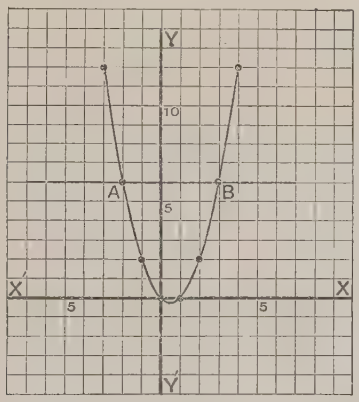


Fig. 28.

The graph, as shown in Fig. 28, cuts $y = 6$ at the points A and B whose abscissae are -2 and $+3$ respectively.

$\therefore$ required solution is $x = -2$ or $+3$.

Note i. When the graph $y = x^2 - x$ is drawn we can at once solve any equation of the form $x^2 - x - c = 0$ where c is a number, e. g.

The roots of $x^2 - x - 4 = 0$ are $x = 2{\cdot}6$ or $-1{\cdot}6$ since the graph cuts $y = 4$ at points where $x = 2{\cdot}6$ or $-1{\cdot}6$.

Note ii. Since the graph does not go lower than $y = -{\cdot}2$ (approximately), it is seen that if c is less than $-{\cdot}5$, there are no real values for x. An accurately drawn graph on a larger scale would show exactly the least value of y, but with this graph we are not justified in saying more than that the least value of y is about $-{\cdot}2$ or $-{\cdot}3$.

147. Graphical Method III.

Solve $x^2-x-6=0$.

What value of x makes $x^2=x+6$?

Draw the graphs of $y=x^2$ and $y=x+6$, and find the abscissae of the points of intersection.

(i) Graph of $y=x^2$:

x	-3	-2	-1	0	1	2	3	4
y	9	4	1	0	1	4	9	16

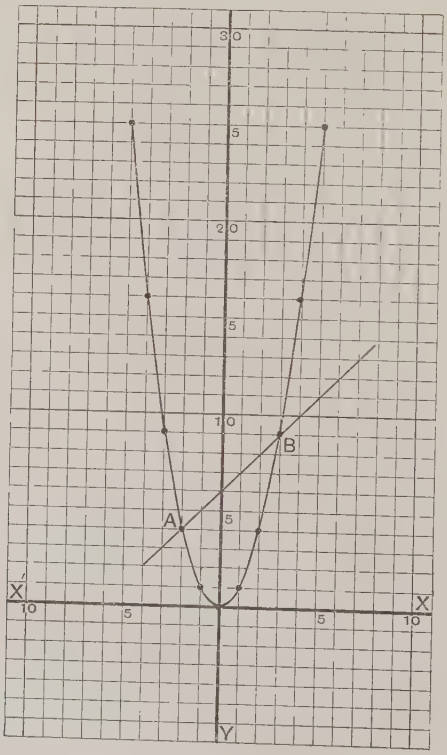


FIG. 29.

(ii) Graph of $y = x + 6$.

The equation is of first degree, $\therefore$ the graph is a straight line.

x	0	-3	$+3$
y	6	3	$+9$

Fig. 29 shows that the two graphs cut at the two points A and B, where $x = -2$, $x = +3$ respectively.

Hence required solution is $x = -2$ or $+3$.

Note. The graph of $y = x^2$ should be drawn accurately on a fairly large scale, and can then be used for solving any quadratic equation. The straight line need not be actually drawn, a piece of thread can be stretched across the graph of $y = x^2$, and the abscissae of the points of intersection can easily be read.

Or a piece of cardboard or celluloid can be cut to the shape of the graph $y = x^2$; with this the graph can be reproduced whenever required.

Examples XVII b.

Solve the following equations by each of the three graphical methods, the answers being correct to one decimal place:

1. $x^2 - 4x - 5 = 0$.
2. $x^2 - 3x - 5 = 0$.
3. $x^2 + 6x + 8 = 0$.
4. $x^2 + 9x + 3 = 0$.
5. $x^2 - 5x + 8 = 0$.
6. $x^2 - 12x = -36$.
7. $(x - 3)(x + 4) = 15$.
8. $4x^2 - 5x - 6 = 0$.
9. $3x^2 - 11x + 14 = 0$.
10. $7x^2 - 2x - 1 = 0$.

By drawing graphs, find approximately at least one root of each of the following equations:

11. $x^3 + 2x^2 - 3x + 4 = 0$.
12. $x^3 - 3x + 2 = 0$.
13. $x^3 - 7x^2 + 3 = 0$.

Solve by one of the graphical methods as correctly as you can:

14. $x^2 - 2{\cdot}3x + 1{\cdot}1 = 0$.
15. $x^2 + 3{\cdot}4x - 2{\cdot}3 = 0$.
16. $x^2 - 4{\cdot}1x = 6{\cdot}7$.
17. $3x^2 - 11x + 7{\cdot}2 = 0$.

ELEMENTARY THEORY OF QUADRATICS

148. Every quadratic equation has two roots and not more than two different roots.

Let $ax^2 + bx + c = 0$ be the quadratic equation. It is shown in

§ 144 that this equation has two roots. If possible, let the equation have three different roots, α, β, γ, then

$$a\alpha^2+b\alpha+c=0, \quad \text{(i)}$$

$$a\beta^2+b\beta+c=0, \quad \text{(ii)}$$

$$a\gamma^2+b\gamma+c=0. \quad \text{(iii)}$$

Subtract ii from i $\quad a(\alpha^2-\beta^2)+b(\alpha-\beta)=0.$

We may divide by $\alpha-\beta$ since α and β are supposed different;

$$\therefore \quad a(\alpha+\beta)+b=0. \quad \text{(iv)}$$

Similarly, by subtracting iii from ii, we obtain

$$a(\beta+\gamma)+b=0. \quad \text{(v)}$$

Subtract v from iv $\quad a(\alpha-\gamma)=0;$

but $\alpha-\gamma$ is not zero, $\quad \therefore \quad a=0.$

Substitute in iv $\quad b=0.$

Substitute in i $\quad c=0.$

Hence the equation cannot be satisfied by three different values unless all the coefficients are zero; in that case the equation is an identity.

Example. *Prove the identity*

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$$

The left-hand side is a quadratic expression; the equation is satisfied by $x=a$, $x=b$, $x=c$, and the equation is, therefore, an identity.

149. The sum of the roots of any quadratic equation is equal to the coefficient of x with the sign changed, divided by the coefficient of x^2; and the product of the roots is equal to the constant term divided by the coefficient of x^2.

Method I.

Consider the standard form

$$ax^2+bx+c=0.$$

Divide by a $\quad x^2+\frac{b}{a}x+\frac{c}{a}=0.$

To solve we try to resolve this into factors; in order to resolve into factors, we try to find two quantities whose product is $\frac{c}{a}$ and whose algebraic sum is $\frac{b}{a}$. Suppose h and k are these quantities,

then $$(x+h)(x+k) = 0;$$

$$\therefore \quad x = -h \quad \text{or} \quad -k;$$

and sum of roots $$= -(h+k) = -\frac{b}{a},$$

product of roots $$= hk = \frac{c}{a}.$$

Method II.

By the formula of § 144 the roots of

$$ax^2+bx+c = 0$$

are given by $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Hence, if α and β are the roots,

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a};$$

$$\therefore \quad \alpha+\beta = \frac{-2b}{2a} = -\frac{b}{a},$$

$$\alpha\beta = \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a} = \frac{c}{a}.$$

If the quadratic equation be written

$$x^2+px+q = 0,$$

we have $$\alpha+\beta = -p, \ \alpha\beta = q.$$

By means of these relations between the roots, questions can often be answered without actually solving the equation.

Example. *Find the sum of the squares of the roots of*

$$x^2-17x-19 = 0.$$

Let α and β be the roots, then

$$\alpha+\beta = 17,$$
$$\alpha\beta = -19.$$
$$\therefore\ \alpha^2+\beta^2 = (\alpha^2+2\alpha\beta+\beta^2)-2\alpha\beta$$
$$= 17^2-(-38)$$
$$= 289+38$$
$$= 327.$$

150. Nature of the roots of a quadratic equation

By the formula the roots of $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

(i) If $b^2-4ac > 0$, then $\sqrt{b^2-4ac}$ is a real quantity, and the roots are real and unequal.

(a) If b^2-4ac is an exact square, $\sqrt{b^2-4ac}$ is rational, and the roots are real, unequal, and rational.

(b) If b^2-4ac is not an exact square, $\sqrt{b^2-4ac}$ is irrational, and the roots are real, unequal, and irrational.

(ii) If $b^2-4ac=0$, then $\sqrt{b^2-4ac}=0$, and the roots are real and equal.

(iii) If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is imaginary, and the roots are imaginary and unequal.

These rules are very useful, as they enable us to choose the best method of solving an equation. If, for instance, mental work shows us that b^2-4ac is negative, it is no use to try a graph method, and there is often no need to solve the equation at all; if b^2-4ac is not an exact square, it is no use to try to solve by the factor method, and the formula method or a graph method should be adopted at once.

151. Equal roots. The student may have some difficulty in understanding what is meant by equal roots. If, for instance, we have to solve $x^2-6x+9=0$, what do we gain by saying that there are two roots equal to 3 instead of that there is one root equal to 3? It is quite clear that

there are two factors $x-3$, so that we are justified in saying that there are two roots, but what is the use of this idea of two equal roots? The answer to this depends on the original question from which the equation was derived, e. g. find where the graphs represented by the two equations $x^2+y^2=18$, $x+y=6$ meet.

It has been shown in § 64 that, in order to find where two curves meet, we must treat their equations as simultaneous equations and find the resulting values of x and y.

$$x^2+y^2=18, \qquad \text{(i)}$$

$$x+y=6. \qquad \text{(ii)}$$

From ii $\quad y=6-x.$

Substitute in ii then $\quad x^2+(6-x)^2=18;$

i.e. $\quad 2x^2-12x+36=18,$

i.e. $\quad x^2-6x+9=0, \qquad \text{(iii)}$

i.e. $\quad x=3$ or $3.$

Substitute in ii $\quad y=3$ or $3.$

Hence the two curves meet twice at the point (3, 3).

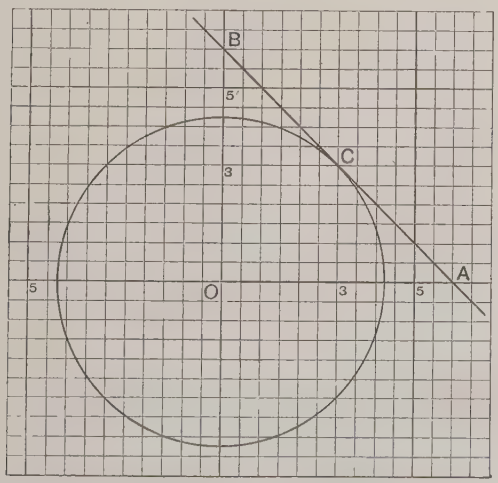


Fig. 30. Scale: 2 sides = 1 unit.

The two curves are shown in Fig. 30, where the circle is the graph of $x^2+y^2=18$ and the straight line AB is the graph of $x+y=6$. It is seen that the line AB just meets the circle at the point C where $x=3$, $y=3$, but it does not cut the circle. When two curves or lines meet but do not cut they are said to touch. This is sometimes expressed thus :

Two lines, whether straight or curved, which meet in two coincident points are said to touch one another. We see that since the equation iii has two equal roots, the graphs of equations i and ii touch one another.

Examples XVII c.

If α and β are the roots of $x^2+px+q=0$, find the values of :

1. $\alpha-\beta$. 2. $\alpha^3-\beta^3$. 3. $\alpha^3-\beta^3$. 4. $\alpha^3+\beta^3$.

Without actually solving, find (i) the sum of the roots, (ii) the difference of the roots, (iii) the sum of the squares of the roots of each of the following equations :

5. $x^2-7x+8=0$. 6. $3x^2-4x+16=0$.
7. $2x^2-3x-5=0$. 8. $x^2+21x+6=0$.
9. $x^2-2ax+c=0$. 10. $x^2+2ax+a^2=b^2$.

Form the simplest equations whose roots are

11. $2+\sqrt{5}$, $2-\sqrt{5}$. 12. 3 and -8.
13. $3-\sqrt{-7}$, $3+\sqrt{-7}$. 14. $-2+\sqrt{13}$, $-2-\sqrt{13}$.

15. Find the simplest rational equation that is satisfied by $x=-4-\sqrt{5}$. What are the other roots of the equation ?

16. Show that the roots of $x^2-(1+k)x+k=0$ are rational.

17. Prove that the roots of $2x^2-2(1+k)x+k=0$ are real provided k is real.

18. By solving the equation $x^2-4x+5=k$, show that if x is real x^2-4x+5 cannot be less than 1.

19. Draw the graph of $y=x^2-3x+4$. Find algebraically where it is cut by the graph of $y=mx$. For what values of m are these two points coincident ? Interpret your result geometrically.

Examples XVII.

Solve the following equations:

1. $4x - \frac{14-x}{x+1} = 14.$
2. $3x - \frac{3x-10}{9-2x} = 2 + \frac{6x^2-40}{2x-1}.$
3. $\frac{x}{5+x} + \frac{7}{6-4x} = \frac{11x}{11x-8}.$
4. $x^4 - 10x^2 + 9 = 0.$
5. $x - 4\sqrt{x} - 5 = 0$, and verify.
6. $\frac{x-1}{x+1} + \frac{x+3}{x-3} = 2\left(\frac{x+2}{x-2}\right).$
7. $x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6.$
8. $x^2 + \frac{1}{x^2} + 2x + \frac{2}{x} = 33.$
9. $\frac{7}{x^2+4x} + \frac{21}{3x^2-8x} = \frac{22}{x}.$
10. $(x^2-4x)^2 - 6(x^2-4x) + 5 = 0.$
11. $px^2 - qx + r = 0.$
12. $\frac{2x}{x-1} + \frac{3x-1}{x+2} - \frac{5x-11}{x-2} = 0.$
13. $x(x+b) + a(a-b) = 2ax.$
14. $(a-b)x^2 - (c-a)x + (b-c) = 0.$
15. $\frac{1+2x-3x^2}{1-2x+3x^2} = \frac{3-2x+x^2}{3+2x-x^2}.$
16. $(2x^2+3x-1)(2x^2+3x-2) = 156.$
17. $x(x+3) + a(a-3) = 2(ax-1).$
18. $\frac{5x+4}{5x-4} + \frac{5x-4}{5x+4} = 2\frac{1}{6}.$
19. $(x+1)^4 - (x-1)^4 = 14x.$
20. $156x - x^2 = 5184.$

By graphical methods obtain the roots of the following equations correct to two decimal places:

21. $4x^2 - 5x + 1 = 0.$
22. $3x^2 - 5x = 6.$
23. $3x^2 - 4x - 2 = 0.$

24. $3x^2 - 2{\cdot}34\,x = 3{\cdot}45$.

25. $x + \dfrac{1}{x} = 3$.

26. One of the roots of the equation $2x^3 - 9x^2 + 4x + 12 = 0$ is $x = 2$. Find the others.

27. Prove that, if either of the roots of a quadratic with rational coefficients is rational, they are both rational. Form the quadratic, one of whose roots is $-3 + \sqrt{5}$.

28. Prove that $6x^2 - 29x + 35$ is positive when $x = 2\frac{1}{4}$, negative when $x = 2\frac{5}{12}$, and again positive when $x = 2\frac{2}{3}$. For what values of x does it vanish? Illustrate by a graph.

29. The sum of a certain quantity and its reciprocal is 2·05. Find it.

30. If one root of a quadratic equation is known, state a simple way of finding the second root.

Solve $x(x-b) = a(a-b)$.

31. Write down the equation whose roots are 3, -4, $\frac{2}{5}$; and reduce it to its simplest form.

32. Prove that if x is real $x^2 + 4x + 11$ cannot be less than 7.

33. The difference of the roots of the equation $x^2 + ax + 6 = 0$ is 6. Find the value of a to two decimal places.

34. Find an expression which exceeds $2x^2 + 17x - \frac{1}{3}$ by the quantity by which it is exceeded by $340x^2 - 19x - 605$. For what values does the expression vanish?

35. What quantity may be substituted for a in $(a-3)(a-4)$ without altering its value?

36. By drawing graphs of x^2 and of $3x + 1$, find approximately the roots of $x^2 - 3x - 1 = 0$.

37. Without solving the equation $x^2 - 3x - 5 = 0$, find the sum of the cubes of the roots.

38. If α and β are the roots of $x^2 - 5x + 18 = 0$, find the equation whose roots are 3α and 3β.

39. By drawing a graph, or otherwise, find the least possible value for $x^2 + x - 2$, if x is a real quantity.

40. Draw the graphs of $x^2 + x - 2$ and of $3x + 6$. What equation can be solved by the use of these two graphs? What are its roots?

41. If $x^2 + 3x + 4 = y$, what value of y will give equal roots for x? Illustrate by a graph.

42. Draw the graph of $y = x^2 - 3x + 4$. What is the value of m if the graph of $y = mx$ touches the preceding graph?

43. A regiment of 1000 men is drawn up in oblong form so that the difference of the numbers in the two adjacent sides is 117. How many men were there in the long side of the oblong?

44. A farmer bought a certain number of oxen for £240, and after losing 3 sold the remainder for £8 a head more than they cost him, thus gaining £59. How many did he buy?

45. A and B distribute £60 each among a certain number of persons. A relieves 40 persons more than B, but B gives to each 5*s.* more than A. How many persons did B relieve?

46. A person bought for one crown as many pounds of sugar as were equal to half the number of crowns he spent. In selling the sugar he received for every 25 lb. of sugar as many crowns as the whole had cost him, and received on the whole 20 crowns. How many crowns did he spend, and what did he give for a pound?

47. A person bought some sheep for £72, and found that if he had bought 6 more for the same money he would have paid £1 less for each. How many did he buy?

48. A labourer dug two trenches, one of which was six yards longer than the other, for £17. 16*s.*, and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

49. A company at a restaurant had £8. 15*s.* to pay. Two left without paying their share, and each of those who were left had 10*s.* more to pay to make good the deficit. How many were there at first?

50. A regiment can be formed into a hollow square, three deep. With the addition of 25 men a solid square can be formed; the number of men in each side of the solid square is greater by 22 than the square root of the number of men in each side of the hollow square. How many men are there in the regiment?

CHAPTER XVIII

GRAPHS OF EQUATIONS OF THE SECOND AND HIGHER DEGREE. APPLICATIONS

152. The graph of any algebraic function of a single variable can be obtained by plotting a number of points as explained in § 47. In this chapter we shall give examples of the graphs that commonly occur, and shall show how the labour of plotting a large number of points may in some cases be considerably lessened.

153. It has been pointed out in § 49 that the graph of an equation of the first degree is a straight line. Care must be taken

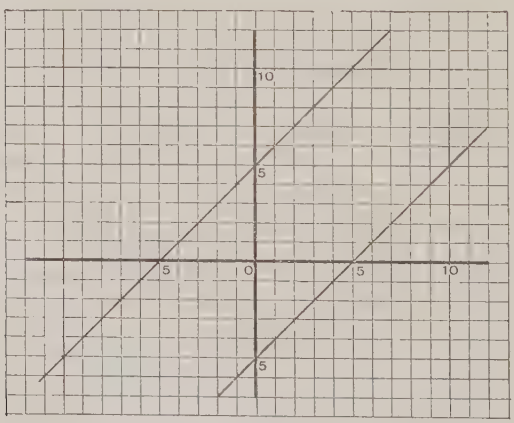


Fig. 81.

that the equation is cleared of all algebraical fractions before its degree is ascertained; for instance,

$$y = \frac{2}{x} + 3, \quad y - 3 = \frac{4}{x-5},$$

are not of the first degree and the graphs are not straight lines.

If, when all the terms of an equation are transposed to the left-hand side, the left-hand side can be resolved into a number of factors of the first degree, the graph consists of an equal number of straight lines.

Example. *Draw the graph of* $x^2-2xy+y^2=25$.

Equation is $(x-y)^2-5^2=0$;

i. e. $(x-y-5)(x-y+5)=0$.

This equation is satisfied if either $x-y-5=0$, or $x-y+5=0$.

The graph, therefore, consists of the graphs of these two equations, that is, of the two straight lines shown in Fig. 31.

154. Equations of second degree.

Type I. $y=ax^2$.

Notice (i) That the graph passes through the origin.

(ii) That x^2 must be positive Hence, if a is positive, y is

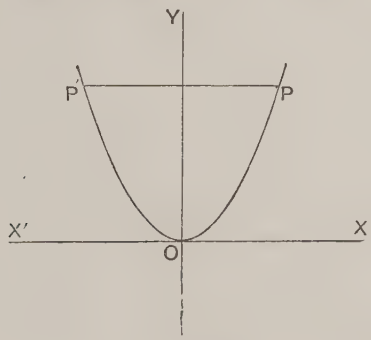


Fig. 32.

positive; if a is negative, y is negative; the curve, therefore, lies entirely above or entirely below the axis of x.

(iii) That y is unchanged, if the sign of x is changed. If the graph passes through a certain point P, it will, therefore, pass through another point P' such that PP' is parallel to the axis of x and bisected by the axis of y. We express this by saying that the graph is symmetrical with respect to the

axis of y. We have already had many examples of this graph which is of the form shown in Fig. 32. This curve is called a **parabola.**

Example. *Draw the graph of* $y = -3x^2$.

(i) The graph passes through the origin.

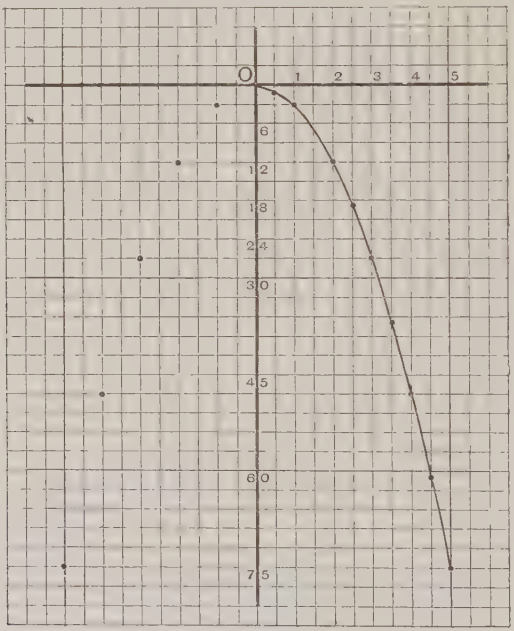


FIG. 33. Scale : 2 sides = 1 for x; 1 side = 3 for y.

(ii) Since x^2 is positive y must be negative, and the curve is entirely below the axis of x.

(iii) Since there are no odd powers of x, the graph is symmetrical with regard to the y-axis.

x	0	1	2	3	4	5	·5	2·5	3·5	4·5
x^2	0	1	4	9	16	25	·25	6·25	12·25	20·25
y	0	−3	−12	−27	−48	−75	−·75	−18·75	−36·75	−60·75

Since y soon gets large compared with x, a smaller scale must be used for y as shown in Fig. 33.

When the first six points are plotted, it is seen to be advisable to plot some intermediate points before drawing the curve. When the portion of the graph corresponding to positive values of x has been drawn, the portion corresponding to negative values can be filled in by plotting as many symmetrical points as are necessary, no further calculation is required.

The point where the parabola crosses the line of symmetry is called the *vertex*.

155. Type I (extended). $y = ax^2+bx+c$.

Several equations of this form have already occurred in connexion with the graphical solution of quadratic equations. The following way of dealing with them is instructive.

To draw the graph of $y = 3x^2-6x+5$.

Note that the graph is not symmetrical with regard to either axis and does not pass through the origin.

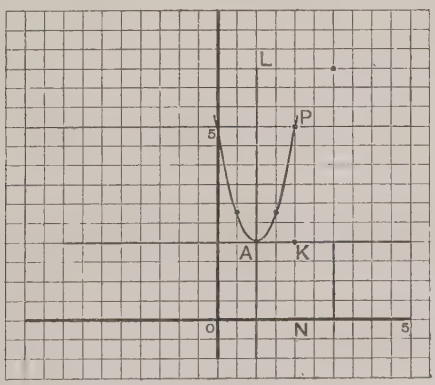


Fig. 34. Scale: 2 sides = 1 for x and y.

Rearrange the equation as follows:

$$y = 3(x^2-2x)+5 \qquad \text{(i)}$$
$$= 3(x-1)^2+2,$$
$$\text{or} \quad (y-2) = 3(x-1)^2. \qquad \text{(ii)}$$

In Fig. 34 A is the point $x = 1$, $y = 2$; P is any point on the

required graph so that the length of PN is y and of ON x. It follows that the length of PK is $y-2$ and of AK $x-1$. Hence equation ii asserts that $PK = 3AK^2$. That is, if A had been the origin and AK the axis of x, the equation of the curve would have been $y = 3x^2$.

But this is the form discussed in the preceding section, therefore the curve is a parabola with A as vertex and AL as line of symmetry.

In the same way it may be shown that the graph of any equation of the form $y = ax^2+bx+c$ is a parabola.

In actually plotting the graph, it is better to use the old method; the method of this section is used when one wishes to avoid actually drawing the curve.

Examples XVIII a.

Without actually drawing the graphs, state which of the following equations represent parabolas, and give all the information you can about the parabolas.

1. $y = -x^2$. 2. $y = 3x^2-5$. 3. $x = 4y^2$.
4. $x^2-y^2 = 4$. 5. $x-2y^2 = 4y$. 6. $4x-3 = 2x-3y^2$.
7. $(x-2)^2 = 3y^2-8y$ 8. $y^2 = 4x$. 9. $y^2+4x = 3$.

10–18. Draw the graphs of Nos. 1–9.

156. Type II. $xy = a$.

Take as an example $xy = 6$.

(i) The graph is not a straight line since xy is of two dimensions, nor does it pass through the origin.

x	0	± 1	± 2	± 3	± 4	± 5	± 6	± 12
y	∞	± 6	± 3	± 2	$\pm 1{\cdot}5$	$\pm 1{\cdot}2$	± 1	$\pm{\cdot}5$

Since xy is positive, like signs must be taken together.

If $x = 0$, y equals the result of dividing 6 by nothing, an operation which cannot be performed. Consider what y becomes as x decreases.

x	·5	·25	·2	·1	·01	·001
y	12	24	30	60	600	6000

As x approaches the value nothing, $\frac{6}{x}$ continually increases, and can be made greater than any number that can be thought of, by taking x sufficiently small. Such a large number is called **infinity** and is denoted by the symbol ∞. The same argument can be

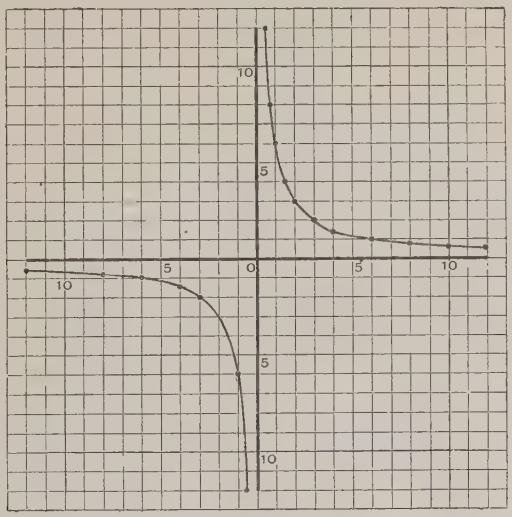


FIG. 35.

used for any number a. The result is sometimes stated concisely as follows: when $x = 0$, $\frac{a}{x} = \infty$.*

The graph of $xy = 6$ is shown in Fig. 35. This curve is known as the **hyperbola**. It consists of two distinct branches; and both branches get continually nearer and nearer to the axes but never actually meet them.

* Up to the present we have always used the symbol 0 to denote absolutely nothing, that is, the difference between two exactly equal numbers. It is here used to denote a very small number, so small that for practical purposes it may be regarded as nothing. The statement given above really means 'when x is very small, $\frac{a}{x}$ is very large.'

A straight line to which a curve gets continually nearer and nearer but which it never actually meets is called an **asymptote** of the curve.

157. Type II (extended). $xy+ax+by+c=0.$

Take as an example $xy-2x+3y-18=0.$

Factorize the variable part, adding whatever constant is necessary to complete the factorization.

$$x(y-2)+3(y-2)=12;$$

i.e. $$(x+3)(y-2)=12.$$

Then by the same kind of argument as for Type I (extended),

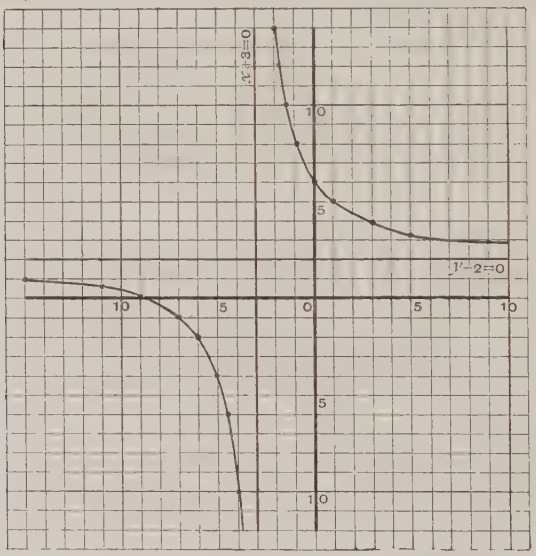


FIG. 36.

this is seen to be the equation of a hyperbola having the graphs of $(x+3)=0$ and $(y-2)=0$ for asymptotes. The graph should then be plotted as follows:

x	-3	-2	-1.5	-1	0	1	3	5	9	∞
$(x+3)$	0	1	1.5	2	3	4	6	8	12	∞
$(y-2)$	∞	12	8	6	4	3	2	1.5	1	0
y	∞	14	10	8	6	5	4	3.5	3	2

There are a similar set of negative values of $(x+3)$ and $(y+2)$ which are easily plotted by treating the asymptotes as axes and the values of $x+3$ and $y-2$ as co-ordinates.

The graph is shown in Fig. 36.

This graph can also be drawn by the following method:

$$xy-2x+3y-18=0.$$

Rearrange $$y(x+3)=2x+18$$

$$y=\frac{2(x+3)+12}{x+3}$$

$$=2+\frac{12}{x+3}.$$

Tabulate in the usual way:

x	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
$\frac{12}{x+3}$	$-2{\cdot}4$	-3	-4	-6	-12	∞	12	6	4	3	$2{\cdot}4$
y	-4	-1	-2	-4	-10	∞	14	8	6	5	$4{\cdot}4$

If $x=-3-$ a small quantity, $y=-$ a large quantity;

if $x=-3+$ a small quantity, $y=+$ a large quantity.

Examples XVIII b.

Trace the graphs of the following equations:

1. $xy=-6$.
2. $x(y-3)=9$.
3. $(x-1)(y+2)=-8$.
4. $xy-3x-4y=0$.
5. $3xy=(x-1)(y-2)$.
6. $2x+5xy=34$.
7. $(x-3)(y-2)=0$.
8. $(x-3)(y-2)=1$.
9. $(2x-3y+1)(3x-4y+2)=0$.
10. $(3x+2y-1)(3x-2y-2)=0$.
11. $2x^2+xy-y^2-3y-2=0$.
12. $6x^2-5xy-4y^2+4x-9y-2=0$.
13. $y=\frac{1}{3}\left\{\frac{30}{x-5}+23-2x\right\}$ from $x=0$ to $x=5$. Find where $y=2x$.

158. Equation of second degree.

Type III. $x^2+y^2=a^2$.

Consider $x^2+y^2=25$.

(i) Since no odd powers occur, the curve is symmetrical with regard to both axes; only positive values need be calculated, the other values being filled in by symmetry.

(ii) Since $y^2 = 25 - x^2$, and y^2 cannot be negative,

$\therefore$ x cannot be greater than 5 or less than -5.

Similarly y cannot be greater than 5 or less than -5.

x	0	1	2	3	4	5
x^2	0	1	4	9	16	25
$25 - x^2$	25	24	21	16	9	0
y	5	4·9	4·6	4	3	0

(iii) Also since x and y can be interchanged in the equation

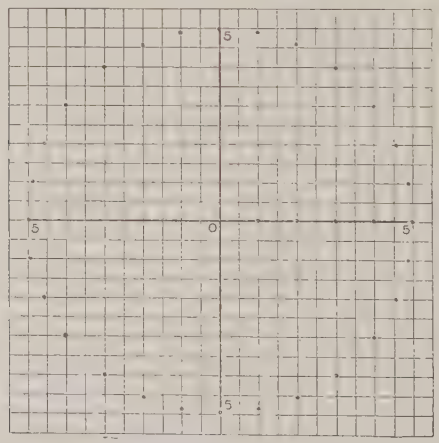


FIG. 37. Scale: side of square = .5.

without altering the equation, we may interchange the values of x and y, thus obtaining $x = 4{\cdot}9$, $y = 1$ and $x = 4{\cdot}6$, $y = 2$, &c.

These values are plotted in Fig. 37 and the symmetrical points also are shown, thus giving 28 points on the graph. It is obvious that the graph is a circle with radius 5 and centre at the origin. By using compasses the graph can be drawn accurately.

159. Equations of second degree, Type III (extended).

$$x^2+y^2+ax+by+c=0.$$

By using Pythagoras' Theorem, viz. 'The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the sides containing the right angle', we can prove that the graph of an equation of the form

$$x^2+y^2+ax+by+c=0$$

is a circle and can show how the centre and radius may be determined.

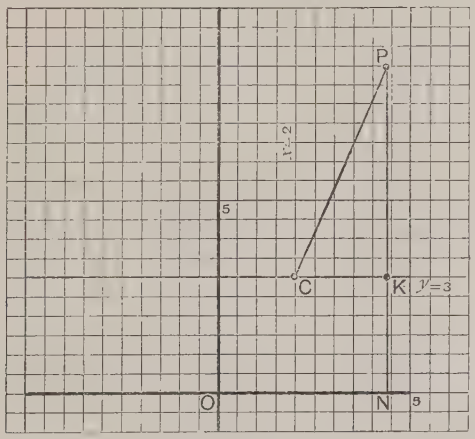


Fig. 38. Scale : side of square = ·5.

Take as an example $x^2+y^2-4x-6y-23=0$. (i)

Rearrange $x^2-4x+y^2-6y=23.$

Complete the two squares by adding $4+9$ to each side,

$$(x-2)^2+(y-3)^2=36. \quad \text{(A)}$$

In Fig. 38 the graphs of $x=2$ and $y=3$ are drawn; C is the point (2, 3), and P is any point on the graph of equation (i).

Then the length of PN is y and of PK $y-3$;

also, the length of ON is x and of CK $x-2$.

Hence equation A gives the geometrical statement,

$$CK^2+PK^2=36.$$

But by Pythagoras' Theorem $CK^2 + PK^2 = CP^2$;

$$\therefore \quad CP^2 = 36, \quad \text{i.e. } CP = 6.$$

Hence the point P, i.e. any point on the graph, is at a constant distance 6 from the fixed point C.

That is, the graph of equation (i) is a circle with centre (2, 3) and radius 6.

In the same way it can always be shown that **If the coefficients of x^2 and y^2 are the same both in magnitude and sign, and there is no term containing xy, then the graph of the equation is a circle.**

160. Equation of second degree.

Form V $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Example I. $\quad \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

No odd powers occur, therefore the graph is symmetrical with regard to both axes and we need calculate only positive values.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}, \quad \text{but } \frac{y^2}{4} \text{ cannot be negative;}$$

$$\therefore \quad \frac{x^2}{9} \not< 1;$$

$$\therefore \quad x \not< 3 \text{ and } \not> -3.$$

Similarly $\quad y \not< 2$ and $\not> -2$.

x	0	1	2	3	2·6	2·9
$\frac{x^2}{9}$		·11	·44	1	·75	·94
$1 - \frac{x^2}{9}$	1	·89	·55	0	·25	·06
y^2	4	3·55	2·22	0	1	·25
y	2	1·9	1·5	0	1	·5

Plotting these points and the symmetrical points, the graph enclosed in the rectangle *PQRS* (Fig. 39) is obtained.

This curve is called an ellipse.

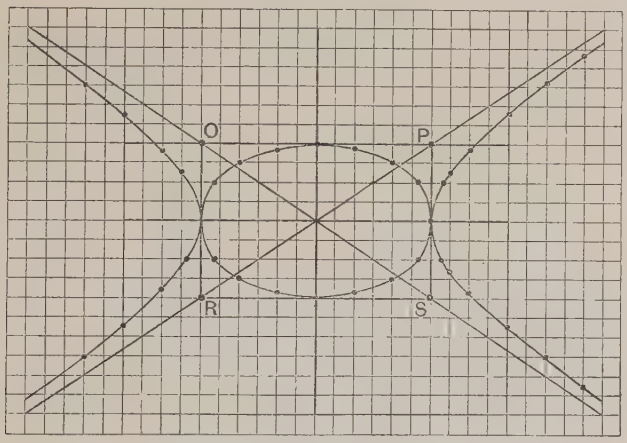


Fig. 39. Scale : side of square = .5.

Example II. $\frac{x^2}{9} - \frac{y^2}{4} = 1.$

No odd powers occur, therefore the graph is symmetrical with respect to both axes.

$$\frac{x^2}{9} = 1 + \frac{y^2}{4} \quad (y \text{ may have any value});$$

$$\frac{y^2}{4} = \frac{x^2}{9} - 1 \quad (x^2 \text{ cannot be less than } 9).$$

x	3	4	5	6	7	8	9	10	3·5	3·3	3·1
$\frac{x^2}{9} - 1$	0	·77	1·77	3	4·44	6·11	8	10·11	·36	·25	·06
y^2	0	3·11	7·11	12	17·77	24·44	32	40·44	1·44	1	·25
y	0	1·8	2·7	3·5	4·2	4·9	5·7	6·4	1·2	1	·5

These points and the symmetrical points are also plotted in Fig. 39. The graph is seen to be a hyperbola having the diagonals of the rectangle $PQRS$ as asymptotes.

Examples XVIII c.

Draw the graphs of the following equations:

1. $x^2 = y^2 - 2(y^2 - 72)$.
2. $(x-3)^2 + (y+4)^2 = 49$.
3. $x^2 - 2x + y^2 + 6y = 26$.
4. $(3-x)^2 + (y-3)^2 = 9$.
5. $x^2 + y^2 - 4x + 4y + 8 = 0$.
6. $x^2 - y^2 = 16$.
7. $9x^2 + 16y^2 = 144$.
8. $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{16} = 1$.
9. $\dfrac{(x+2)^2}{16} - \dfrac{(y-2)^2}{9} = 1$.
10. $x^2 - y^2 + x + 3y - 2 = 0$.

Solve graphically:

11. $x^2 + y^2 = 25$, $x + y = 1$.
12. $x^2 + y^2 = 10$, $x + 4y = 7$.
13. $x^2 + y^2 - 6x + 8y = 11$, $x^2 + y^2 = 9$.
14. $x^2 + y^2 - 4x + 6y = 12$, $4x + 3y + 26 = 0$.
15. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$, $y - 2x = 1$.

161. Graph of equation of third degree.

The graphs of equations of degrees higher than the second must be obtained by ordinary plotting. The graph of $y = x^3$ should be drawn carefully; it is useful and is a good example of one type of cubic curve (i. e. graph of equation of third degree). We give here as an example the graph of a more complicated equation.

Example. *Draw the graph of* $y = x^3 - 2x^2 - x + 2$.

x	-3	-2	-1	0	1	2	3	-1.5	$-.5$	$.5$	1.5	2.5
x^3	-27	-8	-1	0	1	8	27	-3.37	$-.13$	$.13$	3.37	15.6
$-2x^2$	-18	-8	-2	0	-2	-8	-18	-4.5	$-.5$	$-.5$	-4.5	-12.5
$-x$	$+3$	$+2$	$+1$	0	-1	-2	-3	$+1.5$	$+.5$	$-.5$	-1.5	-2.5
y	-40	-12	0	2	0	0	8	-4.37	1.87	1.13	$-.63$	2.6

This graph is shown in Fig. 40. Notice that it has two turning points A and B.

Applications of Graphs.

162. Examples have already occurred showing how graphs may be used to solve simple simultaneous equations

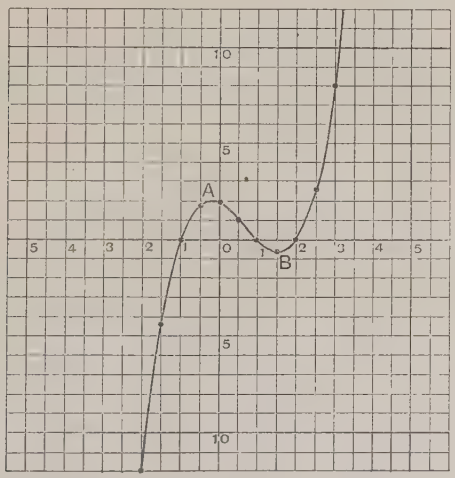


FIG. 40. Scale: 2 sides = 1 for x; 1 side = 1 for y.

and quadratic equations. Similar methods may be employed to solve equations of higher degree.

Example I. *Solve* $x^3-2x^2-x-2=0$.

This may be solved by finding where the graph of

$$y=x^3-2x^2-x-2$$

cuts the axis of x. Fig. 40 shows that the required roots are

$$x=-1,\ 1,\ \text{or}\ 2.$$

Example II. *Solve* $x^3-3x^2+2x+4=0$.

The method of Example I might be employed, or the following method.

Since the equation is equivalent to

$$x^3 = 3x^2 - 2x - 4,$$

we can solve by finding where the graphs of

$$y = x^3 \text{ and } y = 3x^2 - 2x + 4$$

intersect.

These graphs are shown in Fig. 41, where it is seen that the only real solution is $x = -.8$.

Example III. *Solve* $x^2 + y^2 - 6x + 4y = 16,$ (i)

$$xy - 3x + 2y = 6.$$ (ii)

Draw the graphs of these two equations and find the co-ordinates of their points of intersection.

Equation i is $(x-3)^2 + (y+2)^2 = 29.$

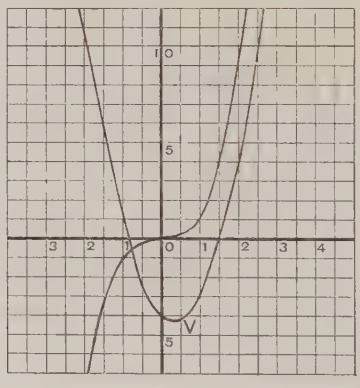


Fig. 41.

The graph is a circle whose centre is $(3, -2)$ and radius $\sqrt{29}$. This square root should be found by geometry rather than by arithmetic. This is easily done since $29 = 5^2 + 2^2$.

Equation ii is $(x+2)(y-3) = 0.$

The graph consists of the two straight lines whose equations are $x = -2$, $y = 3$ respectively.

163. Maximum and minimum values.

By means of graphs, maximum and minimum values can

be approximately found which can be found accurately only with the aid of the Differential Calculus.

Example I. *Find the minimum value of* $3x^2-2x-4$.

The graph of $y = 3x^2-2x-4$ is the parabola shown in Fig. 41.

It is seen that the value of y is least at the turning point V. At the point V, $x = \cdot3$, $y = -4\cdot3$.

Hence $3x^2-2x-4$ is least when $x = \cdot3$, and its value is then $-4\cdot3$.

With a larger scale, these values can be found more accurately.

Another method. Quadratic functions can be dealt with as follows.

$$\begin{aligned} 3x^2-2x-4 &= 3\{x^2-\tfrac{2}{3}x \quad\}-4 \\ &= 3\{x^2-\tfrac{2}{3}x+\tfrac{1}{9}\}-4-\tfrac{1}{3} \\ &= 3\,(x-\tfrac{1}{3})^2-4\tfrac{1}{3}\,; \end{aligned}$$

but a square quantity cannot be less than nothing. Hence the right-hand side is least when $(x-\frac{1}{3})^2 = 0$, and the required minimum value is $-4\frac{1}{3}$.

Example II. *Find the maximum and minimum values of* x^3-2x^2-x+2.

Fig. 40 shows that this function continually increases as x increases from $-\infty$ until x is $-\cdot2$ approximately, the value then begins to diminish. Hence, when $x = -\cdot2$, the function is a maximum, and its value is then 2·1.

As x increases from $-\cdot2$ to 1·5 the function continually decreases and afterwards increases. Hence, when $x = 1\cdot5$, the function has a minimum value, its value then being $-\cdot6$.

Notice that maximum and minimum values are not necessarily the same as greatest and least values. In this case the greatest and least values are respectively $+\infty$ and $-\infty$.

A maximum value of a function of x is a value which is greater than the values obtained when x is either slightly diminished or slightly increased; and a minimum value is one which is less than the values obtained when x is either slightly diminished or slightly increased.

Examples XVIII d.

Solve graphically:

1. $y = -0{\cdot}4x^2 + 1{\cdot}2x - 0{\cdot}8$, $y = 0{\cdot}5x - 1{\cdot}1$.
2. $y = {\cdot}7x^2 - 2{\cdot}2x + 1{\cdot}8$, $y = {\cdot}3x - {\cdot}4$.
3. $x^3 - 2x = \frac{x^2}{4} + \frac{1}{2}$.
4. $y = 2x^2 - 6x + 3$, $y = x^2 - 2x - 1$.
5. What value of y will give equal roots for the equation
$$x^2 - 3x + 4 = y?$$
6. $x^3 - 10{\cdot}1x^2 + 25{\cdot}4x = 0$.
7. $y = x + 2 + \frac{1}{x}$, $y = x^2 + 2x + 1$.
8. $x^3 + 2x^2 - x - 1 = 0$.
9. $x^4 - 4x = 8$. Verify by drawing graphs of $y = x^3$, $y^2 - 4x = 8$
10. $x^3 + 4{\cdot}84x + 2 = 0$.
11. Find graphically the maximum and minimum values of
$$\frac{x^2 + 6}{2x + 1}.$$
12. Find the least value of $x^2 + x - 2$ (i) algebraically, (ii) graphically.
13. Find graphically and algebraically the maximum or minimum value of $3 + 6x - x^2$.
14. Find the maximum and minimum values of
$$x^3 - 2x^2 - 5x + 3.$$
15. Divide the number 20 into two parts, such that the square of one together with three times the square of the other shall be a minimum.
16. A number is added to 2·25 times its reciprocal; for what number is the sum a minimum?
17. Draw the graphs of $y = 3 - 2x$ and $y = \frac{3}{x + 3}$ for values of x from -2 to $+3$, and hence find the positive root of
$$2x^2 + 3x - 6 = 0.$$

164. Graphs may be used instead of tables for reduction purposes, or to find square roots, &c.

Example I. *Draw a graph by means of which English money may be reduced to French money; it being given that* £1 = 25·3 francs.

Let £x be equivalent to y francs,
then $25{\cdot}3x = y$.
The graph is therefore a straight line.
When $x = 0$, $y = 0$; when $x = 2$, $y = 50{\cdot}6$.

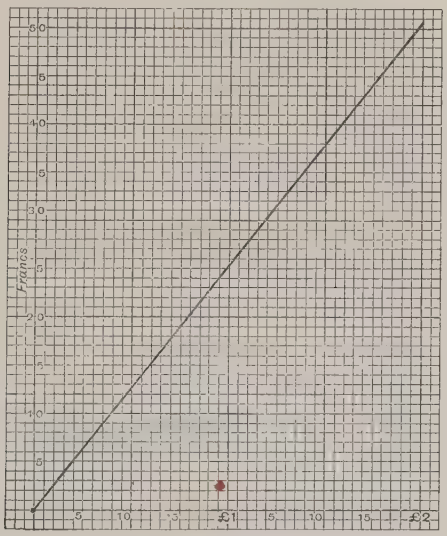


Fig. 42.

Fig. 42 shows the graph, and by means of it any sum less than £2 may be expressed in francs; and any sum less than 50 francs may be expressed in shillings.

Example II. *Draw a graph by means of which the cube roots of numbers from* 0 *to* 10 *may be found approximately.*

Here $y = \sqrt[3]{x}$, and graph must be drawn by plotting.
We have $x = y^3$.

y	1	2	3	1·5	2·5	2·25	·5	·8
x	1	8	27	3·4	15·6	11·4	·13	·5

The graph is shown on a small scale in Fig. 43; from it we see that $\sqrt[3]{2} = 1{\cdot}22$, $\sqrt[3]{4{\cdot}8} = 1{\cdot}68$, &c.

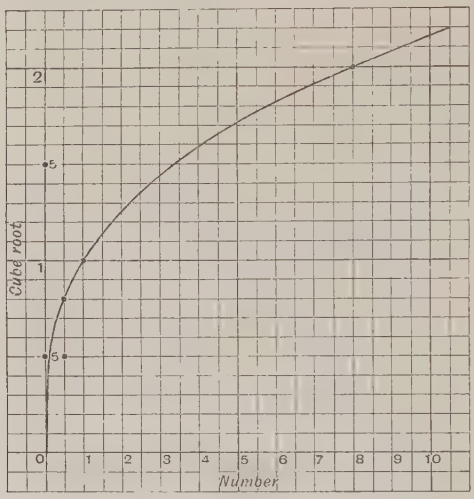


Fig. 43.

165. When observations are plotted, the resulting graph may suggest a law connecting the quantities under observation. If, for instance, the graph looks like a parabola, the law may be of the form $y = a + bx + cx^2$. By substituting the observed values, this suggested formula may be tested and possibly found to be true.

Example.

Simultaneous values of x *and* y *are given in the following table:*

x	0	1	2	3	4	5	6	7	8
y	3·25	3·45	3·65	5·0	6·45	8·25	10·45	13·0	16·0

Plot these values and endeavour to find a simple relation between x *and* y.

The graph looks like a parabola with its vertex at (0, 3·25).
Assume therefore as equation

$$y = 3{\cdot}25 + ax^2.$$

When $x = 1$, $y = 3{\cdot}45$; $\therefore \quad 3{\cdot}45 = 3{\cdot}25 + a$,

i.e. $a = {\cdot}2$.

We have, therefore, as the suggested relation

$$y = 3{\cdot}25 + {\cdot}2x^2.$$

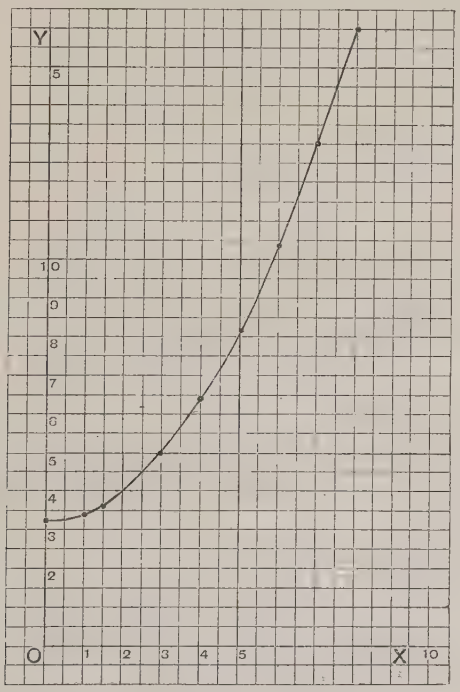


FIG. 44.

Test this by substituting the other values for x.

When $x = 3$, $3{\cdot}25 + {\cdot}2x^2 = 5{\cdot}05$.
When $x = 4$, $3{\cdot}25 + {\cdot}2x^2 = 6{\cdot}45$.
When $x = 5$, $3{\cdot}25 + {\cdot}2x^2 = 8{\cdot}25$.
When $x = 6$, $3{\cdot}25 + {\cdot}2x^2 = 10{\cdot}45$.
When $x = 7$, $3{\cdot}25 + {\cdot}2x^2 = 13{\cdot}05$.
When $x = 8$, $3{\cdot}25 + {\cdot}2x^2 = 16{\cdot}05$.

Examples XVIII e.

1. Draw a graph by means of which the square roots of numbers from 0 to 5 may be found to two decimal places. Use it to find $\sqrt{3{\cdot}6}$, $(1{\cdot}71)^2$.

2. Make a graph to reduce inches to centimetres; given 39·4 centimetres = 36 inches.

3. Freezing-point on a Fahrenheit thermometer is 32°, on a Centigrade thermometer 0°; the boiling points are 212°, 100° respectively. Draw a graph by means of which any Centigrade reading between 0 and 100 may be expressed in the Fahrenheit scale?

What is the equation of the graph?

4. Plot the following points:

x	-3	-1	3	4
y	-7	-1	11	14

If the points lie on a straight line find its equation.

5. When a weight W is tied to the end of a vertical elastic cord, the following observations of the length l are made:

W	300	500	1000	1500	1700
l	98·7	100·2	104·4	108·3	110·4

It is known that the graph obtained by plotting the values ought to be a straight line; find (i) the probable length of the cord when no weight is attached, (ii) which of the five observations is most inaccurate.

6. Two quantities, U and C, are found to have the following simultaneous values:

U	0·67	1·00	1·366	1·46	2·49
C	4·84	6·25	8·60	9·11	14·25

Is there any simple approximately correct law connecting U and C?

7. Draw a graph to show the simple interest on £10 at $3\frac{1}{2}$ per cent. for any number of days less than 101 days.

Hence find the interest on £15 for 39 days at $3\frac{1}{2}$ per cent.

8. The following simultaneous values of two quantities, c and s are known. Find a simple relation connecting c and s.

c	1	·98	·94	·77	·71	·57	·42	·10
s	0	·17	·34	·64	·71	·82	·91	·99

9. Find a simple relation connecting x and y as given by the following table:

x	2	3	6	9	14	17
y	5	3·33	1·67	1·11	·71	·59

10. The time t secs. that a pendulum of length l feet takes to oscillate is given in the following table:

l	6	5	4	3	2	1
t	2·7	2·46	2·2	1·9	1·56	1·1

Show, by plotting the values of l and t^2, that a relation of the form $t = \sqrt{cl}$ exists, and find c from the graph.

Examples XVIII.

1. Draw graphs to represent the two expressions $4x+5$ and x^2; and thence obtain the solutions of the equation $x^2 = 4x+5$.

2. Plot the graph $y = x^2-8x+15$ between the values $x = -1$ and $x = 6$, thus obtaining the minimum value of y.

3. Plot the following values of x and y, and hence find the values of x when $y = ·89$ and of y when $x = ·8$.

x	0	·3	·7	·9	1·3	1·6	2	2·4	2·6	3
y	0	·29	·64	·78	·96	1	·91	·68	·52	·14

4. Solve by a graphical method $x^2+3x = 2$.

5. Solve by a graphical method

$$y-5 = 2(x-7),$$
$$y^2 = 21+4x-x^2.$$

6. A volume of gas occupies a volume v when the pressure is p; the following sets of simultaneous values of p and v are observed. Determine whether any simple relation connects p and v; and if so find it.

p	50	76	110	135	150	200	240
v	15·2	10	6·9	5·6	5·1	3·8	3·1

7. Plot the graph of $(y-1)^2 = 8(x+2)$.

8. Draw the graphs of $y = \frac{x}{3} + \frac{20}{3x}$ and $y = 3x-9$, and hence solve the equations $x^2-3xy+20 = 0$, $y-3x+9 = 0$.

9. In an examination the top boy obtains 215, the bottom boy 96; it is required to change the marks so that the top boy may

receive 90 and the bottom boy 25. Draw a graph by means of which the new marks of the intermediate boys may be found. [First show algebraically that the graph is a straight line.]

10. Draw a graph by which the cube root of any number between 0 and 10 may be found. By means of it find

$$\sqrt[3]{4{\cdot}2},\quad \sqrt[3]{7},\quad \sqrt[3]{9{\cdot}6}.$$

11. Draw the graph of $y = \dfrac{(1+2x)(2+x)}{x}$, and hence show that the right-hand side can never lie between 1 and 9.

12. Solve $x+y=7$, $xy=11$, and verify by drawing the graphs of the equations.

13. Draw the graph passing through the following points; find the equation of the graph:

x	0	·1	·3	·8	1	1·7
y	0	5·72	17·2	45·8	57·3	97·4

14. Draw the graph of $y = x + \dfrac{1}{x+2} + 3$ for values of x between -6 and 2.

15. Solve graphically $x+y=xy$ and $8x+7y=3xy$.

16. When $100y = x^3 - 100x$, calculate the values of y for $x = 8{\cdot}5$, 9, 10, 11, and draw the graph from $x = 8{\cdot}5$ to $x = 11$, showing the scales clearly.

17. Solve graphically $9x^2+4y^2=180$, $xy+12=0$. Verify by an algebraical solution.

18. Show graphically that the values of x and y which satisfy the simultaneous equations $5x = 2y-18$, $5y = 6-7x$, satisfy also the equation $x+y=2$.

19. Solve graphically $2(x-1)(y-1) = 6(x+y) = -3xy$.

20. The following table shows how the volume of a constant weight of water varies at different temperatures. Plot a graph and state what important information can be gathered from the graph.

Temperature	1°	3°	5°	7°	9°
Volume	27·6	25·3	25·2	25·8	27·3

21. By giving a few values to x, show on squared paper the part of the curve $y^2 - 3xy + (1+y)x^2 = 0$ which lies between $x=0$ and $x=1$.

22. A certain function of x is of the form $a+bx+cx^2$, where

a, b, c are constants, such that when x has the values -1, 1, 3 the values of the function are -2, 8, 10 respectively. Determine the values of a, b, and c, and plot a graph of the function between $x = -1$ and $+6$.

23. The distances of an object and its image from a lens, focal length 8 inches, are connected by the relation $\frac{1}{v} - \frac{1}{u} = \frac{1}{8}$, where v inches is the distance of the image and u inches of the object. Draw the graph of this equation, and hence find the value of u when the two distances are equal, and the value of v when $u = \infty$.

24. The following table shows simultaneous values of x and $f(x)$. Draw a graph, and hence find approximately for what values of x $f(x) = 2$.

x	0	·31	·58	·79	·91	1·07	1·24	1·36	1·43
$f(x)$	0	·33	·65	1	1·3	1·80	2·90	4·71	7·12

Graphical Problems.

25. A man starts from a place A to walk at the rate of 4 miles an hour. Draw a graph to show his distance from A at the end of every quarter of an hour. Draw a graph to show that at the end of 10 miles he rested for half an hour and then walked back to A at the rate of 3 miles an hour. From your graph determine the whole time of his walk.

26. Two men start walking towards one another at noon from places 12 miles apart. If one walks at 4 miles an hour, and the other at $3\frac{1}{2}$ miles an hour, determine by a graphical construction when and where they meet.

27. If, in question 26 the faster walker walks half-way and then sits down to wait for the other, find by a graph how long he will have to wait.

28. A sets out from a certain place and travels at the rate of 7 miles in 5 hours; 8 hours afterwards B sets out from the same place and travels the same road at the rate of 5 miles in 3 hours. How long will it take B to overtake A?

29. A and B start at the same time from two places 154 miles apart. A cycles at the rate of 11 miles an hour, B is in a motor travelling at the rate of 23 miles an hour. If they started at noon, when will they meet?

30. A point moves along a straight line so that its distance from one end O at any time x seconds is $x + \frac{10}{x}$ centimetres. Draw a graph showing the distance from O at any time, and find at what time the point is nearest O.

31. If A can make a box in 5 hours, and B a similar box in 4 hours, find by a graphical method how long it will take them to make 6 boxes working together.

32. A tap A can fill a cistern in 4 hours; a tap B can empty it in 7 hours. If both are turned on together how long will it be before the cistern is full?

33. A starts at 3 o'clock walking 4 miles an hour; B follows in the same direction at 3.15 at $4\frac{1}{2}$ miles an hour; then C rides in pursuit, starting at 4 o'clock, at 7 miles an hour. When and where does B catch A, C catch A, C catch B?

CHAPTER XIX

SIMULTANEOUS QUADRATICS

166. If either of two simultaneous equations is of the second degree, they are called simultaneous quadratics. Just as in the case of solving simple simultaneous equations, we try to solve by eliminating one of the unknown letters. This cannot always be done, and sometimes, when it is done, the resulting equation is of a higher degree than the second, and cannot be solved by elementary methods. No method applicable in all cases can be given, but there are two methods that can be used so often that they should be learnt thoroughly.

167. Method I. One of the equations is linear, i. e. of the first degree.

Example I. *Solve* $3x - 2y = 13,$ (i)

$2x^2 + 3y^2 = 30.$ (ii)

From i express one of the letters in terms of the other.

$$3x = 2y + 13.$$

$$\therefore \quad x = \frac{2y+13}{3}. \qquad \text{(iii)}$$

Substitute this value in ii.

$$2\left(\frac{2y+13}{3}\right)^2 + 3y^2 = 30.$$

Simplify $\quad 8y^2 + 104y + 338 + 27y^2 = 270,$

$$35y^2 + 104y + 68 = 0.$$

Solve $\quad (35y + 34)(y + 2) = 0.$

$$\therefore \quad y = -2 \quad \text{or} \quad y = -\tfrac{34}{35}.$$

Substitute in iii.

Then $\quad x = \dfrac{-4+13}{3} \quad \text{or} \quad x = \dfrac{-\frac{68}{35}+13}{3}$

$$= 3 \qquad = \frac{455-68}{3\times 35}$$

$$= \frac{387}{3\times 35}$$

$$= \frac{129}{35}.$$

Solution $\quad x = 3, \quad y = -2;$

or $\quad x = 3\tfrac{24}{35}, \quad y = -\tfrac{34}{35}.$

The answer should not be given

$$x = 3 \text{ or } 3\tfrac{24}{35},\ y = -2 \text{ or } -\tfrac{34}{35},$$

but the corresponding values of x and y should be given together.

This method can be used whenever one of the equations easily gives one unknown in terms of the other.

Example II. *Solve* $3x^2 - 4y^2 = -88,$ (i)

$$2xy = -20. \qquad \text{(ii)}$$

From ii $\quad 2y = -\dfrac{20}{x}. \qquad \text{(iii)}$

Substitute in i $\quad 3x^2 - \dfrac{400}{x^2} = -64.$

Simplify $3x^4+88x^2-400=0,$

$(3x^2+100)(x^2-4)=0.$

$\therefore \quad x^2=4 \quad \text{or} \quad -\frac{100}{3}.$

Extract square root

$x=+2 \text{ or } -2 \text{ or } +10\sqrt{-\tfrac{1}{3}} \text{ or } -10\sqrt{-\tfrac{1}{3}}.$

Substitute in iii

$y=-5 \text{ or } +5 \text{ or } -\frac{1}{\sqrt{-\frac{1}{3}}} \text{ or } +\frac{1}{\sqrt{-\frac{1}{3}}}.$

Hence the real solutions are $x=2, \ y=-5;$

or $x=-2, \ y=+5.$

Example III. *Solve* $3x^2-2x+y^2=37, \qquad (i)$

$xy=-12. \qquad (ii)$

From ii $y=-\frac{12}{x}. \qquad (iii)$

Substitute in i $3x^2-2x+\frac{144}{x^2}=37.$

Simplify $3x^4-2x^3-37x^2+144=0.$

This is of the 4th degree and contains x^3; it cannot be solved by treating it as a quadratic with x^2 as the unknown, as we did in Example II. Our only chance of solving is by using the factor theorem, i.e. we try to find a root by trial.

The possible roots are the factors of 144, viz. 1, 2, 3, 6, 8, 12. &c.

$f(1)=147-39, \quad f(-1)=149-37, \quad f(2)=48-16-148+144,$

$f(-2)=48+16-148+144, \quad f(3)=243-54-333+144=0.$

$\therefore \quad x=3$ is one root and $x-3$ is a factor of the left-hand side.

Hence one solution is $x=3, \ y=-4.$

Divide by $x-3,$

$3x^3(x-3)+7x^2(x-3)-16(x^2-9)=0.$

$\therefore \quad 3x^3+7x^2-16x-48=0.$

By the factor theorem or by the graph method at least one root of this equation can be found; by substitution in iii the corresponding value of y is found.

The factor method can find only rational roots, the graph method will give approximate values of all real roots.

Examples XIX a.

Solve:

1. $2x+3y=8$, $xy=2$.
2. $3x-4y=1$, $3x^2-4y^2=11$.
3. $y-2x=7$, $y^2-4x^2=105$.
4. $3x+y=1$, $9x^2+y^2=61$.
5. $6xy=1$, $(x+1)(y+1)=2$.
6. $x-2y=3$, $x^2-2xy+3y^2=33$.
7. $x^2-xy+y^2=+19$, $x+y=1$.
8. $x^3-xy+y^3=4$, $x+y=1$.
9. $x+ay=1$, $x^2+2ay^2=a^2+1$.
10. $3x-5y=1$, $x^2-2xy-3y^2=-15$.
11. $\dfrac{10x+y}{xy}=3$, $9y-9x=18$.
12. $\dfrac{x+y}{x-y}=\dfrac{13}{5}$, $y^2+x=25$.
13. $x^2+y^2=10$, $x+4y=7$.
14. $5xy+3x-y=492$, $\dfrac{y}{2x}=\dfrac{3}{4}$.
15. $x^2-xy=153$, $x+y=1$.

16. There are two numbers whose difference is $\frac{2}{5}$ of the greater, and the difference of their squares is 128, find them.

17. The difference of two numbers is 15, and half their product equals the cube of the smaller number, find them.

18. The product of two numbers is 56; their difference is 10; find the sum of their squares.

19. A regiment consisting of 1066 men is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each square?

20. A poulterer bought 15 ducks and 12 turkeys for 5 guineas. He had 2 ducks more for 18 shillings than he had turkeys for £1. What was the price of each?

168. Method II.

When every term in the two equations that contains the unknowns is of the same degree, the following rule should

be used:—Eliminate the constant term and factorize the left-hand side of the resulting equation.

Example. *Solve* $x^2-3xy-y^2=9$, (i)

$2x^2+2xy+3y^2=7$. (ii)

Multiply i by 7 $7x^2-21xy-7y^2=63$.

Multiply ii by 9 $18x^2+18xy+27y^2=63$.

Subtract $11x^2+39xy+34y^2=0$.

Resolve into factors $(11x+17y)(x+2y)=0$;

either $11x+17y=0$ or $x+2y=0$,

i.e. $x=-\frac{17}{11}y$ or $x=-2y$.

Substitute in equation i

$\frac{289}{121}y^2+\frac{51}{11}y^2-y^2=9$ or $4y^2+6y^2-y^2=9$,

$(289+561-121)y^2=9\times121$ $9y^2=9$,

$729y^2=9\times121$ $y^2=1$;

$\therefore\ y=+\frac{11}{9}$ or $-\frac{11}{9}$, $\therefore\ y=+1$ or -1,

$x=-\frac{17}{9}$ or $+\frac{17}{9}$; $x=-2$ or $+2$,

each value of x being taken with the value of y above it.

Examples XIX b.

Solve:

1. $3x^2+4y^2=16$, $4xy=8$.
2. $x^2+xy=15$, $xy-y^2=2$.
3. $x^2-2y^2=17$, $x^2+2y^2=33$.
4. $x^2-2xy-y^2=7$, $2x^2-3xy+4y^2=62$.
5. $3x^2-7y^2=5$, $3xy-5y^2=1$.
6. $x^2+xy+y^2=52$, $xy-x^2=8$.
7. $4x^2+9y^2=10$, $2xy+1=0$.
8. $4x^2-xy=14$, $3xy-y^2=5$.
9. $x^2-2xy-y^2=7$, $xy+2y^2=12$.
10. $\frac{x}{y}+\frac{y}{x}=\frac{61}{30}$, $x^2+xy+y^2=91$.
11. Find two numbers such that the sum of their squares is 29 and their product 10.

12. The product of two numbers is 24, and their sum multiplied by their difference is 20, find them.

13. Find two positive integers whose sum multiplied by the greater is 204 and whose difference multiplied by the less is 35.

14. Divide 14 into two integral parts so that the sum of the quotients obtained by dividing each part by the other may be $2\frac{1}{12}$.

15. A man buys tea at two different shops spending altogether £2. 17*s.* 1*d.*; the price in pence per lb. at each shop being equal to the number of pounds bought at that shop. If the price at each shop had equalled as many pence as he bought pounds at the other shop, he would have spent 1*d.* less. How many pounds did he buy at each shop?

169. Miscellaneous Methods.

There are various devices which are useful in solving simultaneous quadratic equations; some of these are shown in the following examples.

Example I. *Solve* $3x+4y = 18,$ (i)

$$xy = 6.$$ (ii)

Square i $\quad 9x^2+24xy+16y^2 = 324.$

Multiply ii by 48 $\quad 48xy \quad = 288.$

Subtract $\quad 9x^2-24xy+16y^2 = 36.$

Extract square root $\quad 3x-4y = +6$ or $-6.$

Add to equation i $\quad 6x \quad = 24$ or $12.$

$\therefore \quad x \quad = 4$ or $2.$

From ii $\quad$ if $x = 4, \quad y = 1\frac{1}{2};$

if $x = 2, \quad y = 3.$

Example II. $x^2-xy = 84,$ (i)

$$xy-y^2 = 48.$$ (ii)

Subtract $\quad x^2-2xy+y^2 = 36.$

Take square root $\quad x-y = +6$ or $-6.$ (iii)

Divide equation i by iii $\quad x = 14$ or $-14.$

Divide equation ii by iii $\quad y = 8$ or $-8.$

Solution $\quad x = 14, \quad y = 8;$

$x = -14, \quad y = -8.$

Example III. *Solve* $2x+5xy = 34,$ (i)

$3y+4xy = 33.$ (ii)

Eliminate xy.

Multiply i by 4 $8x+20xy = 136.$

Multiply ii by 5 $15y+20xy = 165.$

Subtract $8x-15y = -29.$

And solution may be finished by Method I.

Example IV. *Solve* $x^2y^2-5xy+6 = 0,$ (i)

$x^2+2y^2 = 11.$ (ii)

Treat equation i as a quadratic to find xy.

Resolve i into factors $(xy-3)(xy-2) = 0$

$xy = 2$ or $3.$

And the solution may be finished by Method I or II

170. Graphical Method.

Draw the graph of each equation; then the co-ordinates of the points of intersection satisfy both equations, i. e. the co-ordinates of the points of intersection give the required solution (see also § 162).

Example. *Solve* $y = x^2-3x+2,$ (i)

$x = 2y^2-3.$ (ii)

Equation i is not of the first degree, ∴ the graph is not a straight line and at least six points must be found.

x	-3	-2	-1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-3x$	9	6	3	0	-3	-6	-9	-12	-15
y	20	12	6	2	0	0	2	6	12

Equation ii is not of the first degree, ∴ the graph is not a straight line.

y	-3	-2	-1	0	1	2	3	4
$2y^2$	18	8	2	0	2	8	18	32
x	15	5	-1	-3	-1	5	15	29

In Fig. 45 the points of intersection are P and Q, and the solution is approximately $x = \cdot 3$, $y = 1 \cdot 3$ or $x = 2 \cdot 8$, $y = 1 \cdot 6$.

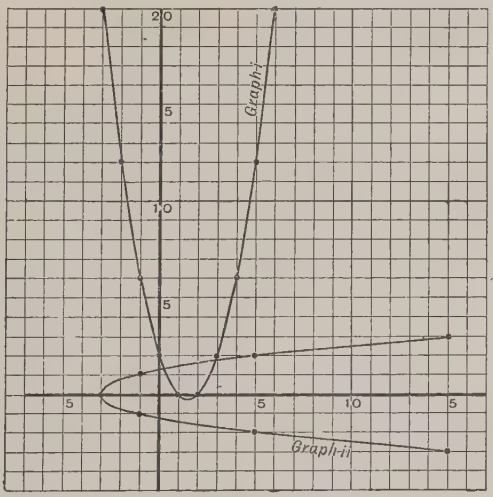


Fig. 45.

To obtain a more accurate solution that portion of the graphs lying between $x = 0$, $x = 3$ should be drawn on a larger scale.

Graph i

x	·1	·2	·3	·4	·5	·6		2·3	2·5	2·6	2·7	2·8	2·9	3
$2+x^2$	2·01	2·04	2·09	2·16	2·25	2·36		7·29	8·25	8·76	9·29	9·84	10·41	11
$3x$	·3	·6	·9	1·2	1·5	1·8		6·9	7·5	7·8	8·1	8·4	8·7	9
y	1·71	1·44	1·19	·96	·75	·56		·39	·75	·96	1·19	1·44	1·71	2

Graph ii

y	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
y^2	1·21	1·44	1·69	1·96	2·25	2·56	2·89	3·24
$2y^2$	2·42	2·88	3·38	3·92	4·50	5·12	5·78	6·48
x	−·58	−·12	·38	·92	1·50	2·12	2·78	3·48

These values are shown plotted in Fig. 46 and the co-ordinates of the point of intersection are there seen to be

$$x = \cdot 26, \quad y = 1 \cdot 28;$$
$$x = 2 \cdot 91, \quad y = 1 \cdot 71.$$

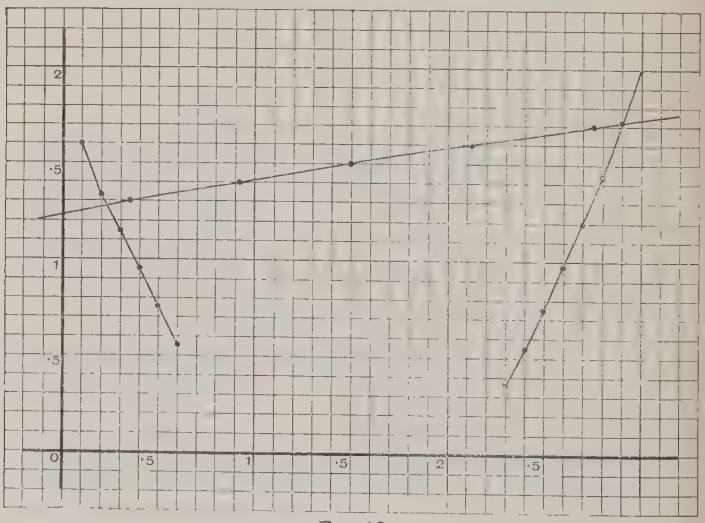


FIG. 46.

On substituting in the original equations it will be found that these are very close approximations to the exact solution.

Examples XIX.

Solve:

1. $x^2 + y^2 = 169,$
$xy = 60.$

2. $9x^2 - 16y^2 = 65,$
$3x - 4y = 13.$

3. $x^3+y^3=35$, $x^2y+xy^2=30$.

4. $x^2-y^2=9$, $2x^2+3y^2=98$.

5. $x^2y^2+16=8xy$, $3x-4y=8$.

6. $xy=20-x^2=y^2-6$.

7. $x^2=y+12$, $y^2=x+12$.

8. $x^2+x+y=18-y^2$, $xy=6$.

9. $x^2+2xy+y^2+2x=120-2y$, $xy-y^2=8$.

10. $x^2+y^2-x-y=78$, $xy+x+y=39$.

11. $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}$, $x-y=2$.

12. $x^2+y^2+x-y=132$, $(x^2+y^2)(x-y)=1220$.

13. $(x-y)^2=8(x+y)$, $(x+y)^2=8(x-y)$.

14. $y^2-3x^2+y+3x+30=0$, $y^2+x^2+y-x-18=0$.

15. $4x^2+5xy=6$, $2y^2-3xy=26$.

16. $\frac{x}{y}+\frac{2y}{x}=3$, $(x+y)(x+2y)=12$

Solve graphically:

17. $y^2-7x-8y+14=0$, $7x+6y=13$.

18. $x-y^2=3y$, $y-3x=0$.

19. $x=3y^2-2y+1$, $3y-4x+5=0$.

20. $x=3y-y^2$, $y^2=4x-2$.

CHAPTER XX

MISCELLANEOUS PROBLEMS

171. Many problems have been set in the previous Example; before giving a set of miscellaneous problems we will work a few to show the kind of way the necessary equations are arrived at. The student is referred to § 109, where some hints have already been given.

Example I. *A merchant gained as many shillings on a certain number of pounds of tea as there were shillings in the cost price of* $10\frac{1}{2}$ *lb. or as there were farthings in the selling price of a* 3 *oz. packet. How many pounds did he sell?*

Let x shillings be selling price of 1 lb., and y shillings be cost price of 1 lb.

Then $x-y$ shillings is gain on 1 lb.

Also $\frac{21}{2}y$ shillings is the cost price of $10\frac{1}{2}$ lb., and $\frac{3}{16}x \times 48$ farthings is the selling price of 3 oz.

$\therefore$ his total gain according to the question is either $\frac{21}{2}y$ shillings or $9x$ shillings. Hence an equation is

$$9x = \tfrac{21}{2}y,$$

$$\text{i.e.} \quad 6x = 7y. \tag{i}$$

Again total gain is $9x$ shillings; gain on one pound is $x-y$ shillings.

$$\therefore \text{ number of pounds} = \frac{9x}{x-y}$$

$$= \frac{9x}{x-\frac{6}{7}x} = 63. \quad \textit{Answer.}$$

This is an example where we do not assume the unknown to be the quantity we wish to find.

Example II. *Two sums of money, making together* £63, *are spent on two different kinds of articles whose prices differ by* 1s., *the number of articles obtained being* 190. *If the two sums are interchanged the number of articles obtained will be* 200. *Find the prices and the sums of money.*

Let x shillings be sum of money spent on articles at y shillings each.

Then $(1260-x)$ shillings is spent on articles at $(y-1)$ shillings each.

$\therefore$ $\frac{x}{y}$ is number of first kind; and $\frac{1260-x}{y-1}$ is number of second kind.

Hence from the question,

$$\frac{x}{y} + \frac{1260-x}{y-1} = 190. \tag{i}$$

If x shillings had been spent on articles at $(y-1)$ shillings each and $(1260-x)$ shillings had been spent on articles at y shillings each, the numbers bought would have been $\frac{x}{y-1}$ and $\frac{1260-x}{y}$ respectively.

Hence $$\frac{x}{y-1}+\frac{1260-x}{y}=200. \qquad \text{(ii)}$$

We now have two equations to find two unknowns.

Add equations i and ii $$\frac{1260}{y}+\frac{1260}{y-1}=390.$$

Multiply by $y(y-1)$ and divide by 30,

$$42(2y-1)=13(y^2-y).$$

Transpose $$13y^2-97y+42=0.$$

Factorize $$(13y-6)(y-7)=0.$$

$$\therefore\ y=7 \quad \text{or} \quad \tfrac{6}{13}.$$

Hence the solution of the equation gives two alternatives, viz. 7*s.* and 6*s.* as the respective prices of an article, or $\frac{6}{13}$ and $-\frac{7}{13}$ shillings as the prices.

A negative price is absurd, therefore the solution $y=\frac{6}{13}$ is rejected.

If $y=7$, we have, by substituting in equation i,

$$\frac{x}{7}+\frac{1260-x}{6}=190.$$

Multiply by 42 $$6x+7\times1260-7x=42\times190.$$

Transpose $$-x=42\times190-42\times210$$

$$=-42\times20.$$

$\therefore$ one sum of money is 42×20 shillings $=$ £42, which is spent on articles at 7*s.* each;

and the other sum of money is £21, which is spent on articles at 6*s.* each.

Note. If $y=\frac{6}{13}$ had been found to be a reasonable solution, the price would have been given not as $\frac{6}{13}$*s.* each, but as 6 shillings for 13.

Example III. *The difference in the rates of two trains is* 13 *miles per hour. The faster of the two takes* 2 *hours less time to travel* 164 *miles than the slower takes to travel* 168 *miles. Find their respective rates.*

Let x miles per hour be the rate of the faster.

Then $\frac{164}{x}$ hours is the time it takes to travel 164 miles.

Also $x-13$ miles per hour is the rate of the slower train.

Then $\frac{168}{x-13}$ hours is the time it takes to travel 168 miles.

From the question this second time is 2 hours more than the first time.

$$\therefore \quad \frac{168}{x-13} - \frac{164}{x} = 2.$$

Multiply by $x(x-13)$,

$$168x - 164x + 164 \times 13 = 2x^2 - 26x.$$

Transpose and divide by 2,

$$x^2 - 15x - 82 \times 13 = 0.$$

Factorize $$(x-41)(x+26) = 0.$$

$$\therefore \quad x = 41 \quad \text{or} \quad -26.$$

The negative value is inconsistent with the meaning of the question. The only meaning we can attach to a rate of -26 miles per hour is that the train is travelling in the opposite direction to some train which is moving with positive velocity; that interpretation does not apply here.

∴ the fast train travels at 41 miles per hour,
the slow at 28 miles per hour.

Example IV. *What is the price of gold per ounce if a rise of 3d. reduces by 5 the number of ounces that can be bought for £5757?*

Let £x be original price per ounce.

Then £$(x + \frac{1}{80})$ is the increased price.

$\frac{5757}{x}$ is the original number of ounces for £5757.

$\frac{5757}{x+\frac{1}{80}}$ is the second number of ounces for £5757.

From the question

$$\frac{5757}{x} - \frac{5757}{x+\frac{1}{80}} = 5.$$

Multiply by $x(x+\frac{1}{80})$,

$$5757 \times \tfrac{1}{80} = 5x^2 + \tfrac{1}{16}x.$$

Multiply by 80 and transpose,

$$400x^2 + 5x - 5757 = 0,$$

$$(80x - 303)(5x + 19) = 0;$$

$$\therefore \quad x = \tfrac{303}{80} \quad \text{or} \quad -\tfrac{19}{5}.$$

The negative value is inconsistent with the question.

∴ required price is £3. 15*s*. 9*d*. per ounce.

172. Algebra can often be used to solve Geometrical questions, e.g.

Example. *Divide a straight line 9 inches long so that the square on one part may be double the square on the other.*

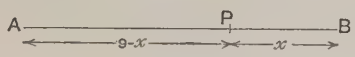


FIG. 47.

Let x inches be the length of the smaller part.
Then $9-x$ inches is length of the larger part.
Then $(9-x)^2 = 2x^2$.
Remove brackets and transpose,

$$x^2+18x-81 = 0.$$

Solve by the formula $x = \dfrac{-18 \pm \sqrt{324+324}}{2}$

$$= \frac{-18+25{\cdot}4}{2} \text{ or } \frac{-18-25{\cdot}4}{2}$$

$$= \frac{7{\cdot}4}{2} \text{ or } -\frac{43{\cdot}4}{2}$$

$$= 3{\cdot}7 \text{ or } -21{\cdot}7.$$

	648	25·4
45	248	
504	2300	

Hence one portion is 3·7 inches long and the other 5·3 inches.

In this case the negative value would mean that BP was to be measured to the right instead of to the left and would give the answer to the question.

A straight line is 9 inches long; produce it so that the square on the whole line, when produced, shall equal twice the square on the part produced.

173. **Problem leading to an indeterminate equation.** Sometimes the data of the question give only one equation although there are two unknowns. In such cases the equation has an infinite number of solutions (§ 60); if, however, the conditions of the question require that the unknowns should be positive integers, the solutions are limited in number, but there may be some difficulty in

finding them. Problems and equations of this kind are said to be **indeterminate.**

Example. *In firing at a target the score is* 5 *for a bull,* 4 *for an inner,* 3 *for a magpie,* 2 *for an outer, and* 0 *for a miss. In how many ways can a score of* 30 *be made with* 7 *shots, it being known that there are no magpies or misses?*

Let x be the number of bulls.
y be the number of inners.
z be the number of outers.

Then
$$5x+4y+2z=30, \quad \text{(i)}$$
$$x+y+z=7; \quad \text{(ii)}$$

so that there are only two equations for three unknowns; but x, y, and z must be positive integers.

Eliminate z by subtracting twice ii from i,

then
$$3x+2y=16. \quad \text{(iii)}$$

Divide by the coefficient of x or y.

Divide by 3
$$x+\tfrac{2}{3}y=5+\tfrac{1}{3}.$$

Transpose integers to the right,
$$\tfrac{2}{3}y-\tfrac{1}{3}=5-x=\text{an integer}.$$

Multiply by an integer k,

then
$$\frac{2ky-k}{3}=\text{an integer}.$$

Choose k so that when $2k$ is divided by 3 the remainder is 1; in this case take $k=2$,

then
$$\frac{4y-2}{3}=\text{an integer},$$

i.e.
$$y+\frac{y-2}{3}=\text{an integer};$$

$$\therefore \quad \frac{y-2}{3}=\text{an integer}=t \text{ (suppose)};$$

then
$$y=3t+2, \text{ where } t \text{ is any integer}.$$

Substitute in iii
$$3x=16-2\times 3t-4;$$
$$\therefore \quad x=4-2t.$$

Substitute in ii
$$z=7-(3t+2)-(4-2t)$$
$$=1-t.$$

Hence if

$t=$	-1	0	1	2
then $x=$	6	4	2	0
$y=$	-1	2	5	8
$z=$	2	1	0	-1

If t is negative, y is negative; if $t>1$, z is negative; and the only possible solutions are when $t = 0$, or 1.

Therefore there are two ways of making a total 30 under the given conditions.

Graphical Method. Draw the graph of equation iii and notice where it passes through corners of squares.

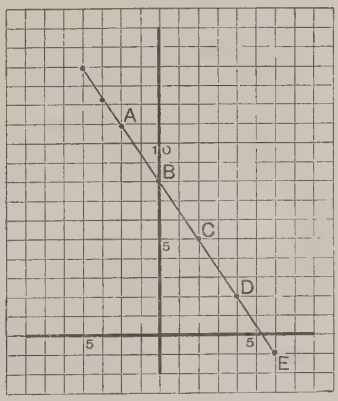


FIG. 48.

Since the equation is of the first degree, the graph is a straight line.

When $x = 0$, $y = 8$;

when $x = -4$, $y = 14$;

when $x = +6$, $y = -1$.

As shown in Fig. 48 the graph passes through the points (0, 8), (2, 5), (4, 2), and these give the only positive integral solutions of equation iii.

174. Questions asking for greatest or least values can often be solved by using the fact that a square quantity cannot be less than zero. They can also be solved by finding the turning-point of a graph.

Example. *What is the area of the biggest rectangular field that can be surrounded by* 400 *yards of fencing?*

Let x yards be the length of one side.

Then $(200 - x)$ yards is the length of the other.

Therefore the area is $x(200-x)$ square yards;

but $$x(200-x) = 200x - x^2$$
$$= -(x^2 - 200x)$$
$$= -(x^2 - 200x + 100^2) + 100^2$$
$$= 100^2 - (x-100)^2;$$

$\therefore$ $x(200-x)$ is greatest when $(x-100)^2$ is least,

i.e. when $(x-100)^2 = 0$,

i.e. when $x = 100$.

Hence greatest area $= 100 \times 100$ square yards
$= 10000$ square yards.

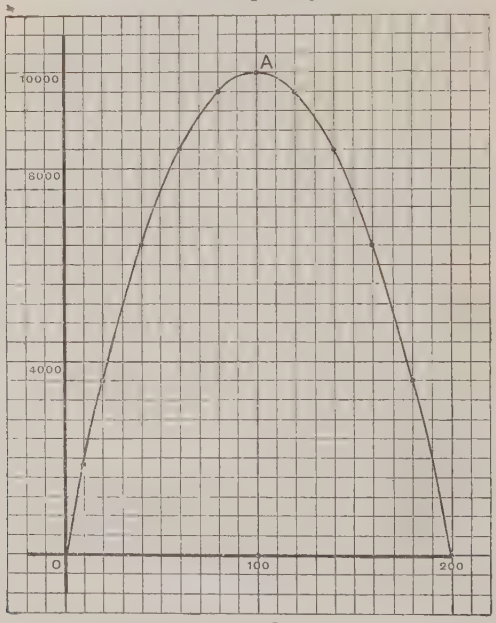


FIG. 49.

Graphical Method. Draw the graph of $y = x(200-x)$. This is shown in Fig. 49.

It is seen that the curve turns at the point A, and that the ordinate y is greater there than at any other point.

Hence the graph shows that y is greatest when $x = 100$, and that its greatest value is 10000.

The graphical method is specially useful when the expression whose maximum or minimum is required is higher than the second degree.

Examples XX.

1. One-fourth of the subscribers to a certain fund gave a sovereign apiece, one-fourth of the remainder gave half a sovereign apiece, and the rest gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half a guinea, and half a crown respectively, the total increase in the subscriptions would be £2. 10*s*. How many subscribers were there, and what was the total amount subscribed?

2. Two workmen take the same time to earn £22 and £21 respectively. The former earns £15. 8*s*. in one day less time than the latter takes to earn the same sum. How much does each earn per day?

3. A man walks 18 miles from A to B in 4 hours and 34 minutes, and does the return journey in 4 hours 50 minutes; walking 4 miles an hour on the level, 3 miles an hour uphill, and 5 miles an hour downhill. Find how many miles there are on the level, how many up, and how many down on his first journey?

4. A draper buys 150 yards of ribbon. Some of this is of fine quality, the rest is of an inferior quality and costs 1*s*. 6*d*. a yard less than the former. If he spends £6 on the dearer ribbon and £15 on the cheaper, what do the ribbons cost per yard respectively?

5. I bought a horse and dog for £25. If I had given 10 per cent. more for the dog and $2\frac{1}{2}$ per cent. more for the horse, I should have spent another 20 shillings. What did I pay for each of them?

6. What is the length of a race in which A beats B by 5 seconds and B beats C by 1 second, the average pace of A being $\frac{1}{3}$ yard per second greater than that of B, and that of B $\frac{1}{7}$ yard per second greater than that of C?

7. A cyclist travels the first half of the distance from P to Q at the rate of 10 miles per hour; he then rests 15 minutes and travels the remainder of the distance at the rate of 15 miles per hour. Had he travelled the whole distance without stopping at the rate of 11 miles per hour he would have taken exactly the same time. Find the distance from P to Q. Verify by a graphical construction.

8. A certain sum of money is to be equally divided among a certain number of people. If the number of people had been greater by ten the share of each would have been smaller by 9*s.* 2*d.*; and if the number of people had been fewer by ten the share of each would have been larger by 13*s.* 4*d.* What was the sum of money to be divided?

9. A fruiterer sold a certain quantity of oranges for £6. 10*s.* If he had given two more oranges for a shilling the same quantity would only have realized £5. 17*s.* How many oranges did he sell?

10. A gravel path two yards wide is made round a square field, and it is found that it takes up one-twentieth of the area of the field. Prove that the area of the field is nearly 25000 square yards.

11. A person buys x articles for y pence and sells them at the rate of y articles for x pence, and he finds that his profit is 30 per cent. almost exactly. What is the value of $\frac{x}{y}$?

12. The perimeter of a rectangle is 340 feet and its diagonal is 130 feet; find the area.

13. The side of a square is 79 feet. Find at what points of the sides the vertices of an inscribed square must be placed in order that it may have an area of 4225 square feet.

14. The prices of seats in the stalls, pit, and gallery of a theatre are respectively seven shillings and sixpence, half a crown, and one shilling. The pit can hold three times and the gallery twice as many people as the stalls. The receipts are £87. 6*s.* when all the seats in the pit and gallery are occupied and all but twelve in the stalls; how many people are present?

15. Rowing at a uniform rate a person can row 5 miles down stream in 30 minutes and back again in 40 minutes; find the velocity of the stream.

16. By adding 30 yards to the length and 10 yards to the breadth of a square field its area is doubled; find its original area in square yards.

17. What number is such that when divided by the product of its digits the quotient is two, and when it is increased by 27 the digits are reversed?

18. A grocer sold 80 pounds of mace and 100 pounds of cloves for £65, but he sold 60 pounds more of cloves for £20 than he did of mace for £10. What was the price of each per pound?

19. The fore-wheel of a carriage makes 6 revolutions more than

the hind-wheel in going 120 yards; but if the circumference of each wheel be increased by one yard it will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each.

20. Find two numbers whose sum multiplied by the sum of their squares is 272, and whose difference multiplied by the difference of their squares is 32.

21. A certain train is 5 minutes late when it performs its journey at the rate of $29\frac{1}{2}$ miles an hour, and it is two minutes late when it travels at 30 miles an hour; what is the length of the journey?

22. In a factory there are employed in one week 57 men, 14 women, and 10 boys, the total wages for the week being £84. 5*s*.; the next week 60 men, 17 women, and 11 boys are employed, the increase of wages amounting to £6. 5*s*. If two women receive as much as a man and a boy, find the weekly earnings of a man, woman, and boy respectively.

23. A person put out a certain sum to interest for $6\frac{1}{2}$ years at 5 per cent. simple interest and found that if he had put out the same sum for 12 years and nine months at 4 per cent. he would have received £185 more. What was the sum invested?

24. An alloy of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

25. A number of two digits when multiplied by the left-hand digit gives product 46; if the sum of the digits be multiplied by the same digit the product is 10. What is the number?

26. A person being asked his age replied, 'If you add the square root of it to half of it and subtract 12 there will remain nothing.' Find his age.

27. *A* and *B* gained by trading £100, which is divided proportionately to the amount of capital each provided. Half of *A*'s capital was £100 less than *B*'s; and *A*'s share of the profits was $\frac{3}{20}$ of *B*'s capital. Find the capital of each.

28. There are two mixtures of whisky and water. *A* contains 30 per cent. of water, *B* 40 per cent. of water. If *A* is poured into B, the resulting mixture contains 36 per cent. of water. If 30 gallons of water are now poured in, the mixture contains 60 per cent. of water. Find the number of gallons in *A* and *B* respectively.

29. A boat which travels at the rate of $10\frac{1}{2}$ miles an hour down

stream takes three times as long to go a certain distance up a river as to go the same distance down. Find the rate of flow of the stream.

30. A train runs a distance of 60 miles. If it had gone 10 miles an hour faster it would have taken half an hour less for the journey. Find the rate of the train.

31. The journey between two towns by one route consists of 233 miles by rail, followed by 126 miles by sea; by another route it consists of 405 miles by rail, followed by 39 miles by sea. If the time occupied on the journey is 50 minutes longer by the first route than by the second, find the average speed by rail, assuming it to be the same by each route and 25 miles an hour faster than the average speed by sea.

32. A and B travel from a town X to a town Y, a distance of $10\frac{1}{4}$ miles. They have a bicycle between them which they can ride at 7 and 10 miles an hour respectively, their walking speeds being $3\frac{1}{2}$ and 4 miles an hour respectively. They start together from X, B walking, A on the bicycle. On arriving at a point P, A leaves his bicycle and walks on. As he is walking into Y he is overtaken by B who has picked up the bicycle and ridden from P. Find the distances of P from X and Y.

33. A working 6 hours a day and B working 7 hours a day together do a piece of work in a certain time. If A worked 7 hours a day and B worked 10 hours a day they could do it in 2 days less. A is twice as rapid a worker as B. Find how many days the work takes.

34. Prove generally that with interest at x per cent. the commercial present worth is less than the true present worth by the fraction $\left(\frac{nx}{100}\right)^2$ of the latter, where n is the number of years.

35. Find the lengths of the two parts into which a line 10 feet long must be divided in order that the areas of the squares on them may be together equal to 64 sq. feet. Give the answer correct to the hundredth of an inch.

36. The hypotenuse of a right-angled triangle is 91 feet long, and the longer of the two remaining sides exceeds the shorter by 49 feet; find the area.

37. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per cubic yard as there were yards in

a side of the other; and the base of the greater exceeded the base of the less by 9 square yards. What was the price of each?

38. The captain of a privateer seeing a trading vessel 7 miles ahead sailed 20 miles in direct pursuit of her; and then, observing that the trader was steering in a direction perpendicular to her former course, changed his own direction so that he overtook the trader without making any further change in his course. The rates of the privateer and trader were 10 miles and 7 miles an hour respectively. What distance did the privateer sail?

39. Divide a straight line 2 inches long into two parts so that the square on one part is double the square on the other part. Account for the two answers.

40. Divide a straight line 3 inches long so that the square on one part equals twice the rectangle contained by the whole and the other part.

41. A person buys two pieces of cloth for £15, the one at 8*s*., the other at 11*s*. per yard, each piece containing more than 10 yards; how many yards did he buy altogether?

42. In how many ways can £1 be paid in half-crowns, shillings, and sixpences, using exactly 18 coins.

43. There are a number of eggs in a basket; when counted 3 at a time there are 2 over; when counted 5 at a time there are 4 over. It is known that there are more than 50 but less than 100 eggs; how many are there?

44. How many pairs of fractions are there, one with denominator 12, the other with denominator 18, whose sum is $\frac{25}{36}$?

45. In a journey of 100 miles a man walked for an exact number of hours at 4 miles per hour and cycled for an exact number of hours at 10 miles per hour, how long did the journey take?

46. A stone is thrown into the air so that its distance from the ground after t secs. is $100\,t - 16\,t^2$, what is the greatest height it reaches? Verify by a graph.

47. A man receives 5 shillings a day while he is engaged on a certain piece of work, but his expenses are half as many shillings per day as he takes days to complete the work; what is the greatest sum he can earn, and how many days will he then take?

48. A cistern without a top and with a square base is to be made with 48 sq. feet of material. What is the maximum content of the cistern?

49. A person stands on the bank of a river which is 100 yards

wide ; he wishes to get to a place A 300 yards along the stream on the opposite bank. He can row 3 miles an hour and run 5 miles an hour along the bank. Whereabouts should he land on the opposite bank so as to reach A as soon as possible ?

50. The combined length and girth of a parcel sent by Parcel Post must not exceed 6 feet. Find the volume of the largest parcel that can be sent. (Assume that the parcel is cylindrical.)

CHAPTER XXI

INDICES

175. For convenience we repeat the definition of x^n.

x^n is the product of n factors each equal to x.

The coefficient 1 is understood, we have therefore adopted the following definition which has several advantages :

x^n is the result of multiplying unity by x n times.

We have also the definition of the n^{th} root, viz. :

$\sqrt[n]{x}$ is a quantity whose n^{th} power is equal to x, i.e.

$$\{\sqrt[n]{x}\}^n = x.$$

With these definitions we proceed to prove the following rules :

176. Multiplication Law.

If two powers of any quantity are multiplied together, the product is a power whose index is the sum of the original indices.

We have to show that $x^p \times x^q = x^{p+q}$.

By definition $x^p = 1 \times x \times x \times x \ldots p$ multiplications.

$x^q = 1 \times x \times x \times x \ldots q$ multiplications.

$\therefore \quad x^p \times x^q = 1 \times x \times x \times x \ldots (p \text{ multiplications}) \times x \times x \times x \ldots$ (q multiplications)

$= 1 \times x \times x \times x \ldots (p+q)$ multiplications

$= x^{p+q}$.

Corollary.

$x^p \times x^q \times x^r = x^{p+q+r}$ and so on for any number of factors.

177. Division Law.

If one power of a quantity be divided by a power of the same quantity, the quotient is a power whose index is obtained by subtracting the index of the divisor from the index of the dividend.

We have to show that $x^p \div x^q = x^{p-q}$.

$$x^p \div x^q = \frac{x^p}{x^q}$$

$$= \frac{1 \times x \times x \times x \ldots p \text{ multiplications}}{1 \times x \times x \times x \ldots q \text{ multiplications}}$$

$$= \frac{x \times x \times x \ldots q \text{ factors} \times x \times x \times x \ldots (p-q) \text{ factors}}{x \times x \times x \ldots q \text{ factors}}$$

$$= 1 \times x \times x \times x \ldots (p-q) \text{ factors}$$

$$= x^{p-q}.$$

178. Involution Law.

If a power of a quantity is itself raised to a power, the result is a power whose index is the product of the two original indices.

We have to show that $(x^p)^q = x^{pq}$.

$$(x^p)^q = 1 \times x^p \times x^p \times x^p \ldots q \text{ multiplications}$$

$$= x^{p+p+p \ldots q \text{ terms}} \quad \text{by § 176}$$

$$= x^{pq}.$$

Corollary

$\{(x^p)^q\}^r = (x^{pq})^r = x^{pqr}$, and so on for any number of successive powers.

179. To show that $(xy)^p = x^p y^p$.

By definition $(xy)^p = 1 \times xy \times xy \times xy \times xy \times \ldots p$ multiplications

$$= 1 \times x \times x \times x \times x \ldots p \text{ multiplications} \times 1 \times y \times y \times y \times y \ldots p \text{ multiplications}$$

$$= x^p y^p.$$

180. Evolution Law.

If any root is taken of any power of a quantity, the result is a power whose index is obtained by dividing the index of the original power by the index of the required root.

We have to show that $\sqrt[q]{x^p} = x^{\frac{p}{q}}$,

$$(x^{\frac{p}{q}})^q = x^{\frac{p}{q} \times q}, \text{ by § 178}$$
$$= x^p.$$

Therefore $x^{\frac{p}{q}} = \sqrt[q]{x^p}$, by definition of root, § 175,

i.e. $\sqrt[q]{x^p} = x^{\frac{p}{q}}$.

181. To show that $\sqrt[q]{x^p} = \{\sqrt[q]{x}\}^p$.

Let $\{\sqrt[q]{x}\}^p = y$.

Then $y^q = \{\sqrt[q]{x}\}^{pq} = [\{\sqrt[q]{x}\}^q]^p = x^p$;

$\therefore \sqrt[q]{x^p} = \sqrt[q]{y^q} = y = (\sqrt[q]{x})^p$.

182. The student must notice carefully what has been assumed in the preceding proofs. In all of them it is assumed that p and q are positive integers, for otherwise the definitions of § 175 have no meaning.

In § 177 it is assumed that p is greater than q.

In § 180 it is assumed that p is a multiple of q, for otherwise $x^{\frac{p}{q}}$ would have no meaning.

Let us now see what results would be obtained if we neglect these restrictions and apply the laws in all cases.

Examples XXI a.

Use the Multiplication, Division, Involution, and Evolution laws for powers to work out the following examples, regardless of all restrictions.

1. $x^3 \times x^5$.	2. $2x^3 \times 3x^5$.	3. $(2x)^3 \times (3x)^5$.
4. $x^7 \div x^2$.	5. $2x^7 \div 3x^2$.	6. $(2x)^7 \div (3x)^5$.
7. $(x^3)^5$.	8. $2(x^3)^5$.	9. $(2x^3)^5$.
10. $\sqrt[3]{x^{12}}$.	11. $8\sqrt[3]{x^{12}}$.	12. $\sqrt[3]{8x^{12}}$.
13. $x^3 \div x^5$.	14. $14x^3 \div 3x^5$.	15. $a^2b^3 \div 2ab^5$.
16. $(2a^3)^4$.	17. $(3a^3)^2$.	18. $(2ab^2)^x$.
19. $\sqrt[3]{x^5}$.	20. $\sqrt[5]{ab^4}$.	21. $\sqrt[4]{a^2b \div a^4b}$.

22. Adopting the definition of § 175, endeavour to obtain a meaning for x^0, y^{-2}.

23. What does $x^{\frac{1}{2}}$ mean according to the definition of x^n? What would $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ mean? Hence deduce the value of $25^{\frac{1}{2}}$.

24. Apply the division law to the following cases:

$$x^3 \div x^3, \quad x^7 \div x^9, \quad x^3y^4 \div x^2y^4.$$

183. It is now seen that if we use the laws of §§ 176–181 in all cases, we obtain powers with negative or fractional indices. We must, therefore, extend the meaning of x^n so as to include cases where n is not a positive integer. This may be done in several ways, all leading to the same conclusions.

184. By division law $x^3 \div x^7 = x^{-4}$.

But $$x^3 \div x^7 = \frac{x^3}{x^7} = \frac{1}{x^4};$$

$\therefore$ x^{-4} must be the same as $\frac{1}{x^4}$.

Similarly $$\frac{x^q}{x^{p+q}} = x^{-p} \text{ by division law,}$$

or $$= \frac{}{x^p} \text{ by dividing throughout by } x^q;$$

$\therefore$ x^{-p} must mean $\frac{1}{x^p}$.

This seems a reasonable interpretation of the meaning of a negative power. For if x^p = (1 multiplied by x p times), it is to be expected that x^{-p} = (1 divided by x p times),

i.e. $$= \frac{1}{x^p}.$$

185. $x^p \div x^p = x^0$ by division law,

but $x^p \div x^p = 1$;

$\therefore$ $x^0 = 1$.

Or from the definition,

x^0 = 1 multiplied by x no times

$= 1$.

186. By evolution law, $$\sqrt[q]{x^p} = x^{\frac{p}{q}}.$$

Hence, if $\frac{p}{q}$ is a fraction, $x^{\frac{p}{q}}$ can mean nothing else than the q^{th} root of x^p, e.g. $x^{\frac{3}{2}} = \sqrt[2]{x^3}$.

It is difficult to show that this agrees with the original definition, but the following example shows how it may be done in special cases.

To find the meaning of $36^{\frac{1}{2}}$.

$36 = 1$ multiplied by 36

$= 1$ multiplied by 6 twice.

Hence, $36^{\frac{1}{2}} = 1$ multiplied by 36 half a time (using def. § 175)

$= 1$ multiplied by 6 once

$= 6$

$= \sqrt{36}$.

Similarly, since $x = 1 \times \sqrt{x} \times \sqrt{x}$, $x^{\frac{1}{2}} = 1 \times \sqrt{x} = \sqrt{x}$.

187. We have now found reasonable meanings to attach to powers whose indices are not positive integers, but in doing so we have assumed two laws, viz. the division and evolution laws, to be true in all cases. It is desirable to assume as little as possible, and we proceed to show how these meanings may be deduced by assuming only one law.

Assume that the multiplication law is true whether the indices are fractional or integral, positive or negative, and accept the results this assumption leads to.*

188. To find a meaning for $x^{\frac{p}{q}}$ where p and q are positive integers.

By the assumption of § 187,

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} = x^{\frac{2p}{q}},$$

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} = x^{\frac{3p}{q}},$$

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times \ldots q \text{ factors} = x^{\frac{p}{q}+\frac{p}{q}+\ldots q \text{ terms}},$$

$$\text{i.e. } \left(x^{\frac{p}{q}}\right)^q = x^{\frac{p}{q} \times q} = x^p;$$

$$\therefore \quad x^{\frac{p}{q}} = \sqrt[q]{x^p}.$$

In numerical examples the result of § 181 is useful, e.g.

$$729^{\frac{5}{6}} = \sqrt[6]{729^5}$$

$$= (\sqrt[6]{729})^5$$

$$= 3^5$$

$$= 243.$$

* Compare § 98.

To find a meaning for x^0.

By the assumption of § 187,

$$x^p \times x^0 = x^{p+0} = x^p.$$

Divide by x^p; $\quad \therefore \quad x^0 = x^p \div x^p = 1.$

To find a meaning for x^{-p} where p is any positive quantity.

By the assumption of § 187,

$$x^{-p} \times x^{+p} = x^{-p+p} = x^0 = 1.$$

Divide by x^p; $\quad \therefore \quad x^{-p} = \dfrac{1}{x^p}$.

189. We are now justified in giving the following definitions:

$x^{\frac{p}{q}}$ means $\sqrt[q]{x^p}$ when p and q are positive integers.

x^{-p} means $\dfrac{1}{x^p}$ when p is any positive quantity.

Examples XXI b.

Find the values of:

1. $8^{\frac{2}{3}}$.
2. 8^0.
3. $8^{-\frac{2}{3}}$.
4. $16^{\frac{3}{4}}$.
5. 16^0.
6. $16^{-\frac{3}{4}}$.
7. $(125)^{\frac{1}{3}}$.
8. $125^{-\frac{1}{3}}$.
9. $64^{\frac{4}{3}} - 36^{\frac{3}{2}}$.
10. $3^3 \times 9^{\frac{1}{2}}$.
11. $4^{\frac{5}{2}} \times 4^{-\frac{1}{2}}$.
12. $3^{\frac{4}{3}} \div 3^{\frac{1}{3}}$.
13. Simplify $\sqrt[3]{5} \times \sqrt{5}$ and $\sqrt[3]{\frac{1}{4}} \times \sqrt[6]{16}$.
14. Prove that $\sqrt[3]{9} \times \sqrt{8} \times \sqrt[6]{36} = 6\sqrt[6]{32}$.

Give alternative forms, not involving negative or fractional indices, for the following expressions:

15. $2x^{\frac{1}{2}}$.
16. $(2x)^{\frac{1}{2}}$.
17. $(2x)^{\frac{1}{2}} \times 2^{\frac{1}{2}} y^{\frac{1}{2}}$.
18. $(3x^{\frac{1}{2}})^3$.
19. $(3x)^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.
20. $x^{-2} y^3$.
21. $9x^{-\frac{1}{2}}$.
22. $2x^{-\frac{1}{2}} \times 3x^{\frac{1}{2}}$.
23. $(3x)^{-\frac{2}{3}} \times \left(\dfrac{x}{3}\right)^{\frac{1}{3}}$.

24. (*a*) Find correct to two places of decimals the values of $10^{\cdot 5}$, $10^{\cdot 25}$, $10^{\cdot 125}$, $10^{\cdot 0625}$. From these values find by multiplication $10^{\cdot 625}$, $10^{\cdot 5625}$, $10^{\cdot 1875}$.

(*b*) Draw the graph of $x = 10^y$ from $x = 1$ to $x = 10$. Hence find the value of y when $10^y = 8$, and what is the value of 10^t when $y = \cdot 48$?

Given that $10^{\cdot 30103} = 2$ and $10^{\cdot 47712} = 3$, find x in the following cases:

25. $10^x = 20$, $10^x = \cdot 2$, $10^x = 1$.

26. $10^x = 300$, $10^x = \cdot 003$, $10^x = \frac{1}{3}$.

27. $10^x = 4$, $10^x = 5$, $10^x = 6$.

28. $10^x = \sqrt[5]{2}$, $10^x = \sqrt[4]{3}$, $10^x = 3^3 \times 2^{10}$.

Between what values must x lie in the following equations:

29. $2^x = 9$, $2^x = \cdot 9$, $2^x = 93$.

30. $10^x = 747{\cdot}2$, $10^x = \cdot 07472$, $10^x = 50$.

190. Before we can use fractional and negative powers, as defined in § 189, we ought to show that they obey the same laws as positive integral powers. We give here the proof of the involution law, and leave the proofs of the others as exercises for the student.

To prove that $(x^p)^q = x^{pq}$ for all values of p and q.

Case I when p and q are both positive integers.
This is proved in § 178.

Case II when q is a positive integer and p has any value.
$(x^p)^q = 1 \times x^p \times x^p \ldots q$ multiplications,
$= x^{pq}$ since the multiplication law is true in all cases.

Case III when q is a positive fraction and p has any value.

Let $q = \frac{h}{k}$, where h and k are positive integers.

Then $$(x^p)^q = (x^p)^{\frac{h}{k}}.$$

Now $$\{(x^p)^{\frac{h}{k}}\}^k = (x^p)^{\frac{h}{k} \times k} \quad \text{(by Case II)}$$
$$= (x^p)^h$$
$$= x^{ph}.$$

Take the k^{th} root of both sides.

Then $$(x^p)^{\frac{h}{k}} = \sqrt[k]{x^{ph}} = x^{\frac{ph}{k}} \quad \text{(§ 180)},$$

i.e. $$(x^p)^q = x^{pq}.$$

Case IV when q is negative and p has any value.
Let $q = -r$, where r is any positive quantity.

Then $$(x^p)^q = (x^p)^{-r} = \frac{1}{(x^p)^r} = \frac{1}{x^{pr}}$$
$$= x^{-pr}$$
$$= x^{pq}.$$

When fractional and negative powers have the meanings derived in § 188, it has now been proved that all the laws

of §§ 176–181 are true. In working examples, therefore, fractional and negative indices are dealt with exactly in the same way as positive indices. The equivalent reciprocal and root forms should generally not be used unless the question requires that the answer be given free of all indices not positive integers, and even then the index forms should be retained as long as possible.

Example I. *Simplify* $\sqrt[4]{x^2y^{-3}z^4} \times (x^{-\frac{1}{2}}y^{\frac{2}{3}}z^{-\frac{5}{6}})^3$.

$$\begin{aligned}\text{Expression} &= x^{\frac{2}{4}}\,y^{-\frac{3}{4}}\,z^{\frac{4}{4}} \times x^{-\frac{1}{2}\times 3}\,y^{\frac{2}{3}\times 3}\,z^{-\frac{5}{6}\times 3} && (\S\ 180)\\ &= x^{\frac{1}{2}}\,y^{-\frac{3}{4}}\,z \times x^{-\frac{3}{2}}\,y^{2}\,z^{-\frac{5}{2}}\\ &= x^{\frac{1}{2}-\frac{3}{2}}\,y^{-\frac{3}{4}+2}\,z^{1-\frac{5}{2}} && (\S\ 176\\ &= x^{-1}\,y^{\frac{5}{4}}\,z^{-\frac{3}{2}}.\end{aligned}$$

If now the answer is required without negative and fractional indices, we have

$$\text{Expression} = \frac{\sqrt[4]{y^5}}{x\sqrt{z^3}}.$$

Example II. *Multiply* $x^{-\frac{1}{3}} - \dfrac{y^{\frac{1}{3}}}{\sqrt[6]{x}} + y^{\frac{2}{3}}$ *by* $\dfrac{1}{x^{\frac{1}{6}}} + \sqrt[3]{y}$.

[Express all roots as fractional powers.]

$$\begin{array}{l} x^{-\frac{1}{3}} - x^{-\frac{1}{6}}\,y^{\frac{1}{3}} + y^{\frac{2}{3}} \\ x^{-\frac{1}{6}} + y^{\frac{1}{3}} \\ \hline x^{-\frac{1}{2}} - x^{-\frac{1}{3}}\,y^{\frac{1}{3}} + x^{-\frac{1}{6}}\,y^{\frac{2}{3}} \\ \qquad + x^{-\frac{1}{3}}\,y^{\frac{1}{3}} - x^{-\frac{1}{6}}\,y^{\frac{2}{3}} + y \\ \hline x^{-\frac{1}{2}} \qquad\qquad\qquad\qquad + y \quad \text{Product.} \end{array}$$

Examples XXI c.

Simplify the following expressions:

1. (i) $(a^{\frac{1}{3}}b^{-\frac{1}{3}})^6 \div (a^{-3}b^{\frac{1}{2}})^{-\frac{3}{2}}$; (ii) $\left(\dfrac{a^{\frac{1}{3}}}{a^{-\frac{1}{3}}}\right)^{\frac{2}{6}}$.

2. $\left(\dfrac{a^{\frac{1}{2}}}{b^{-\frac{1}{3}}}\right)^{\frac{3}{2}} \times \left(\dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{3}}}\right)^{\frac{1}{2}} \div (ab^{-7})^{\frac{1}{12}}$.

3. $\sqrt[5]{\dfrac{a^{\frac{1}{2}} \times b^{-2} \times c^{3}}{a^{-\frac{1}{2}} \times b^{3} \times \sqrt[5]{\frac{1}{c}}}}$.

4. $\left\{\dfrac{a^{\frac{1}{2}}}{b^{-\frac{1}{2}}} \times \dfrac{a^{-1}}{b^{\frac{5}{2}}} \div \dfrac{b^{-\frac{1}{2}}}{a^{\frac{5}{2}}}\right\}^{-24}$.

5. $\left\{(xy^2)^3 \left(\dfrac{x}{y}\right)^4\right\}^3 \div \left\{(x^2y)^4 \left(\dfrac{y}{x}\right)^5\right\}^2$.

6. $\dfrac{4^n \cdot 6^5}{2^3 \cdot 12^n \cdot 3^{-n}}$.

7. $\dfrac{5^3 \cdot 2^{\frac{3}{4}} \cdot 10^{-\frac{1}{4}}}{15^{\frac{3}{4}} \cdot 6^{-\frac{1}{4}} \cdot 4^{\frac{1}{4}}}$.

8. Multiply $a^{\frac{1}{3}} - x^{\frac{1}{3}}$ by $a^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

9. Multiply $x^{\frac{5}{2}}y + x^2y^{\frac{3}{2}} + x^{\frac{3}{2}}y^2 + xy^{\frac{5}{2}}$ by $x^{-1}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{-1}$.

10. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{1}{2}} + y$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

11. Divide $x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{3}{2}} - 2y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

12. If $a = b^{p+1}$ and $b = c^{p-1}$, prove that $ac = c^{p^2}$.

Express as simply as possible:

13. $\sqrt{x} \times \sqrt[3]{x}$.

14. $\sqrt{2} \times \sqrt[3]{4}$.

15. $\sqrt{9a^2b^3}$.

16. $\sqrt[3]{8 \times 17}$.

17. $\sqrt{72}$.

18. $3\sqrt[3]{4} + 2\sqrt[3]{108}$.

19. Simplify $\left(\dfrac{a^{\frac{5}{3}}b^{\frac{1}{3}}c^{\frac{5}{3}}}{a^{\frac{1}{3}}b^{\frac{4}{3}}c^{\frac{7}{3}}}\right)^{-9}$.

20. Simplify $\dfrac{2^{n+3}}{15^{-n-1}} \times \dfrac{6^{-n+2}}{5^{n+1}}$.

Surds.

(The student is advised to revise Examples XII f.)

191. Roots which cannot be found exactly are called surds or irrational quantities; thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[n]{x+y}$ are surds, but $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt{x^2+2xy+y^2}$ are not surds.

An n^{th} root is sometimes called a surd of the n^{th} order.

Many examples involving surds can be worked by using fractional indices, thus:

$$\sqrt[3]{3} \times \sqrt{3} = 3^{\frac{1}{3}} \times 3^{\frac{1}{2}} = 3^{\frac{5}{6}} = \sqrt[6]{243};$$

$$\begin{aligned}\sqrt[3]{3} \times \sqrt{2} &= 3^{\frac{1}{3}} \times 2^{\frac{1}{2}} \\ &= 3^{\frac{2}{6}} \times 2^{\frac{3}{6}} \\ &= (3^2 \times 2^3)^{\frac{1}{6}} \\ &= \sqrt[6]{72}.\end{aligned}$$

192. **The product of the n^{th} roots of two quantities is the n^{th} root of their product.**

For $\sqrt[n]{a} \times \sqrt[n]{b} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}$.

Examples. $\sqrt[3]{7 \times 11} = \sqrt[3]{7} \times \sqrt[3]{11}$, $\sqrt{7 \times 9} = \sqrt{7} \times \sqrt{9} = 3\sqrt{7}$.

The converse proposition is equally useful, viz.

The n^{th} root of the product of two quantities is equal to the product of their n^{th} roots.

Similar rules can be proved for quotients, viz. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and the converse.

Example. *Simplify as far as possible* $\sqrt{12} + \frac{1}{10}\sqrt{75} + 5\sqrt{\frac{1}{12}}$.

Ex. $= \sqrt{4 \times 3} + \frac{1}{10}\sqrt{25 \times 3} + 5\frac{\sqrt{1}}{\sqrt{4 \times 3}} = 2\sqrt{3} + \frac{5}{10}\sqrt{3} + \frac{5}{2\sqrt{3}}$

$= \frac{5}{2}\sqrt{3} + \frac{5}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5}{2}\sqrt{3} + \frac{5}{6}\sqrt{3} = \frac{10}{3}\sqrt{3}$.

193. **The p^{th} root of the q^{th} root of a quantity is the pq^{th} root of the quantity.**

For $\sqrt[p]{\sqrt[q]{a}} = \sqrt[p]{a^{\frac{1}{q}}} = a^{\frac{1}{pq}} = \sqrt[pq]{a}$.

Thus to find the 4^{th} root of a number, we find the square root of the square root; to find the 6^{th} root, find the cube root of the square root.

$$\sqrt[4]{49} = \sqrt{\sqrt{49}} = \sqrt{7}.$$

$$\sqrt[6]{729} = \sqrt[3]{\sqrt{729}} = \sqrt[3]{27} = 3.$$

194. **To reduce a surd to an equivalent surd of another order.**

An example will make the process clear.

Reduce $\sqrt[3]{4x^2y}$ *to a surd of the sixth order.*

$$\sqrt[3]{4x^2y} = 4^{\frac{1}{3}} x^{\frac{2}{3}} y^{\frac{1}{3}}$$

$$= 4^{\frac{2}{6}} x^{\frac{4}{6}} y^{\frac{2}{6}}$$

$$= \sqrt[6]{16x^4y^2}.$$

This shows that we may multiply the order of a surd by any quantity provided we multiply the index of each factor under the radical sign by that quantity.

Examples XXI d.

Simplify:

1. $\sqrt{75}-\sqrt{20}-\sqrt{48}+\sqrt{45}-\sqrt{5}$.
2. $\sqrt{147}-\sqrt{75}+\sqrt{48}$.
3. $\sqrt{50}+2\sqrt{18}-\sqrt{128}$.
4. $\sqrt[3]{24}+\sqrt[4]{768}-\sqrt[5]{96}$.
5. $\sqrt{98}-\sqrt{18}+\dfrac{28}{\sqrt{8}}$.

Find correct to three decimal places:

6. $\sqrt{2}$. 7. $\sqrt[3]{2}$. 8. $\sqrt[4]{2}$.
9. $\sqrt{3}$. 10. $\sqrt[3]{3}$. 11. $\sqrt[4]{3}$.
12. $\dfrac{\sqrt{2}}{2}-\dfrac{1}{\sqrt{2}}$. 13. $(\sqrt{3})^2-(\sqrt{2})^2$.
14. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$. 15. $(\sqrt{3}-\sqrt{2})^2$.
16. $(\sqrt{3}+\sqrt{2})^2$. 17. $(\sqrt{4}-\sqrt{3})^2$.
18. $(2\sqrt{7}-3)(2\sqrt{7}+3)$. 19. $(3\sqrt{5}-4)(3\sqrt{5}+4)$.

Simplify:

20. $(2\sqrt{x}+3\sqrt{y})(3\sqrt{x}+2\sqrt{y})$.
21. $(3\sqrt{a}-4\sqrt{b})(2\sqrt{a}+3\sqrt{b})$.
22. $(\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})$.
23. $(\sqrt{x}+\sqrt{y}+1)(\sqrt{x}+\sqrt{y}-1)$.
24. $(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})$.
25. $(\sqrt{5}+\sqrt{3}-\sqrt{2})(\sqrt{5}-\sqrt{3}+\sqrt{2})$.

What must the following quantities be multiplied by so that the products may be rational? Give the products.

26. $2\sqrt{x}-\sqrt{y}$. 27. $\sqrt{71}$. 28. $3\sqrt{5}+4\sqrt{3}$.
29. $\sqrt[3]{5}$. 30. $2\sqrt{14}-3\sqrt{7}$.

Find the square root of:

31. $a+2\sqrt{ab}+b$. 32. $4+3-4\sqrt{3}$. 33. $3+2\sqrt{2}$.
34. $8-2\sqrt{15}$ 35. $9+4\sqrt{5}$. 36. $26-2\sqrt{133}$.

37. Find the value of $3-2x+x^2$,

when (i) $x=\sqrt{3}$; (ii) $x=\sqrt{3}-\sqrt{2}$,

correct to two decimal places.

38. Find the value, correct to two decimal places, of

$$2+5x-x^2+3x^3-x^4,$$

when (i) $x=2\sqrt{3}$; (ii) $x=\sqrt{3}-1$.

39. Express

$$(\sqrt{8}+\sqrt{2})(\sqrt{8}-\sqrt{2}) \text{ and } (\sqrt{8}+\sqrt{2})^2+(\sqrt{8}-\sqrt{2})^2$$

in their simplest forms.

195. Rationalizing. In numerical work, the denominator of a fraction should always be rationalized before the approximate values of the surds are substituted.

Example I. *Given that* $\sqrt{3}=1{\cdot}732$, $\sqrt{2}=1{\cdot}414$, *find, correct to three significant figures, the value of* $\dfrac{3}{3\sqrt{2}-2\sqrt{3}}$.

$$\frac{3}{3\sqrt{2}-2\sqrt{3}}=\frac{3}{3\sqrt{2}-2\sqrt{3}}\times\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$$

$$=\frac{3(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2})^2-(2\sqrt{3})^2}$$

$$=\frac{3(3\sqrt{2}+2\sqrt{3})}{18-12}$$

$$=\frac{4{\cdot}242+3{\cdot}464}{2}$$

$$=\frac{7{\cdot}706}{2}$$

$$=3{\cdot}85 \text{ correct to 3 significant figures.}$$

The student should work this example by substituting at once for $\sqrt{2}$ and $\sqrt{3}$, and convince himself that the rationalizing method is simpler. A more important reason for rationalizing is this, that the divisor is accurate so that we know exactly how many figures are correct in the answer, whereas in the other method the divisor is approximate.

Example II. *Rationalize the denominator of*

$$\frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}-\sqrt{2}}.$$

$$\text{Fraction} = \frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{5}-(\sqrt{3}-\sqrt{2})}$$

$$= \frac{\sqrt{2}(\sqrt{5}-\sqrt{3}+\sqrt{2})}{5-(5-2\sqrt{6})}$$

$$= \frac{\sqrt{2}(\sqrt{5}-\sqrt{3}+\sqrt{2})}{2\sqrt{6}}$$

$$= \frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{15}+\sqrt{6}-3}{6}.$$

196. Quadratic Surds.

Prop. I. If $\sqrt{a}$ and $\sqrt{b}$ are both surds, and x, a, b are all rational, then $\sqrt{a}$ cannot equal $x+\sqrt{b}$.

Suppose $\sqrt{a} = x+\sqrt{b}$.

Square $a = x^2+2x\sqrt{b}+b$,

whence $\sqrt{b} = \dfrac{a-x^2-b}{2x}$,

or a surd equals a rational quantity, which is impossible;

$$\therefore \sqrt{a} \neq x+\sqrt{b}.$$

Prop. II. If

$$x+\sqrt{a} = y+\sqrt{b}, \text{ then } x = y \text{ and } a = b,$$

where $\sqrt{a}$ and $\sqrt{b}$ are surds and x, y, a, b are all rational.

$$x+\sqrt{a} = y+\sqrt{b}.$$

Transpose $\sqrt{a} = (y-x)+\sqrt{b}$

which is impossible by Prop. I unless $y-x = 0$, in which case $\sqrt{a} = \sqrt{b}$, i.e. $a = b$.

It follows that, if $x+\sqrt{a} = y+\sqrt{b}$ then $x-\sqrt{a} = y-\sqrt{b}$.

Prop. III. If

$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$,

where $\sqrt{b}$ is a surd and a, b, x, y are rational

$$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}.$$

Square $\quad a+\sqrt{b} = x+y+2\sqrt{xy}$;

$$\therefore\ a = x+y$$

$$\sqrt{b} = 2\sqrt{xy}.$$

Subtract $\quad a-\sqrt{b} = x+y-2\sqrt{xy} = (\sqrt{x}-\sqrt{y})^2$.

Take square root $\quad \sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

Example I. *Find the square root of* $5-2\sqrt{6}$.

$$(\sqrt{x}-\sqrt{y})^2 = (x+y)-2\sqrt{xy}.$$

We see, therefore, that we want two numbers whose sum is 5 and product 6. These are 2 and 3.

Hence $\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}$.

Example II. *Find the square root of* $7-4\sqrt{3}$.

$7-4\sqrt{3} = 7-2\sqrt{4\times 3}$ since $4\sqrt{3} = 2\sqrt{4}\times\sqrt{3}$.

$\therefore\ \sqrt{7-4\sqrt{3}} = \sqrt{4}-\sqrt{3} = 2-\sqrt{3}$.

Example III. *Find the square root of* $36+\sqrt{1292}$.

Here the numbers are rather large for mental work, so the following method is advisable.

Let $\quad \sqrt{36+\sqrt{1292}} = \sqrt{x}+\sqrt{y}.$ (i)

Then $\quad \sqrt{36-\sqrt{1292}} = \sqrt{x}-\sqrt{y}.$ Prop. II. § 196

Multiply $\quad \sqrt{36^2-1292} = x-y,$ 216

i. e. $\quad x-y = 2.$ 1296 (ii)

Again, squaring equation i, and equating rational parts

$$x+y = 36. \quad \text{(iii)}$$

From ii and iii $\quad x = 19,\ y = 17$;

$$\therefore\ \sqrt{36+\sqrt{1292}} = \sqrt{19}+\sqrt{17}.$$

197. *Equations involving surds.*

The method of solving is clearly shown by the following examples:

Example I. *Solve* $\sqrt{2x+2}-\sqrt{6x+7}+3=0.$

Transpose so as to obtain a surd by itself on one side.

$$\sqrt{2x+2}+3=\sqrt{6x+7}.$$

Square $2x+2+9+6\sqrt{2x+2}=6x+7.$

Again transpose $6\sqrt{2x+2}=4x-4.$

Divide by 2 and square $18x+18=4x^2-8x+4.$ (A)

Transpose $4x^2-26x-14=0.$

Divide by 2 $2x^2-13x-7=0.$

Factorize $(2x+1)(x-7)=0.$

$$\therefore\ x=7 \text{ or } -\tfrac{1}{2}.$$

Verify. When $x=7$, L. H. S. $=\sqrt{16}-\sqrt{49}+3=4-7+3=0.$

When $x=-\frac{1}{2}$, L. H. S. $=\sqrt{1}-\sqrt{4}+3=1-2+3\neq 0.$

It is seen that $x=-\frac{1}{2}$ does not satisfy if we take the positive root in all cases.

Example II. *Solve* $\sqrt{2x+2}+\sqrt{6x+7}-3=0.$

Transpose as in Ex. I $\sqrt{2x+2}-3=-\sqrt{6x+7}.$

Square $2x+2+9-6\sqrt{2x+2}=6x+7.$

Transpose and divide by 2 $-3\sqrt{2x+2}=2x-2.$

Square $18x+18=4x^2-8x+4.$ (A)

This is the same as equation A in previous example, hence this equation has the same solution, viz. $x=7$ or $-\frac{1}{2}$.

Verify. When $x=7$, L. H. S. $=\sqrt{16}+\sqrt{49}-3=4+7-3\neq 0.$

When $x=-\frac{1}{2}$, L. H. S. $=\sqrt{1}+\sqrt{4}-3=1+2-3=0.$

And in this case $x=-\frac{1}{2}$ satisfies the equation, but $x=7$ does not, if we take the positive root in all cases.

These two examples show that, whenever both sides of an equation are squared in the process of solution, roots may be introduced which do not satisfy the original equation. In all such cases we must substitute in the original equation all the values of x obtained, and so determine which are roots of that equation.

Examples XXI e.

Rationalize the denominators of the following fractions:

1. $\frac{2}{\sqrt{7}}, \frac{3}{\sqrt{5}}, \frac{5}{\sqrt{3}}$.
2. $\frac{\sqrt{2}}{4-\sqrt{8}}$.
3. $\frac{2-\sqrt{3}}{2+\sqrt{3}}$.
4. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
5. $\frac{2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$.
6. $\frac{3x}{\sqrt{a+x}+\sqrt{a-x}}$.

Given that $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, $\sqrt{7} = 2{\cdot}646$, find the values of the following expressions correct to 3 significant figures.

7. $\frac{2}{\sqrt{7}-\sqrt{5}}$.
8. $\frac{2\sqrt{7}+3\sqrt{5}}{3\sqrt{5}-2\sqrt{7}}$.
9. $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+2\sqrt{3}}$.
10. $\frac{1}{\sqrt{7}+2\sqrt{5}} - \frac{1}{\sqrt{7}-2\sqrt{5}}$.
11. $\frac{3}{3\sqrt{3}-5} + \frac{5}{\sqrt{7}+\sqrt{5}}$.
12. $\frac{2\sqrt{5}-\sqrt{7}}{2\sqrt{5}+\sqrt{7}} + \frac{2\sqrt{5}+\sqrt{7}}{2\sqrt{5}-\sqrt{7}}$.
13. $\frac{1}{\sqrt{7}+\sqrt{5}+\sqrt{2}}$.
14. $\frac{\sqrt{5}}{\sqrt{7}-\sqrt{3}+\sqrt{2}}$.

Find the square roots of the following expressions:

15. $8-2\sqrt{15}$.
16. $16+6\sqrt{7}$.
17. $26+\sqrt{660}$.
18. $30-12\sqrt{6}$.
19. $73-12\sqrt{35}$.
20. $151-20\sqrt{57}$.
21. $38-12\sqrt{10}$.
22. $2\frac{1}{4}-\sqrt{5}$.
23. $2a+2\sqrt{a^2-x^2}$

Find correct to three decimal places:

24. $\sqrt{11+6\sqrt{2}}$.
25. $\sqrt[4]{56-24\sqrt{5}}$.
26. $\sqrt[4]{17+12\sqrt{2}}$.
27. Prove that $\sqrt{13+2\sqrt{13}} = \sqrt{\frac{13+3\sqrt{13}}{2}} + \sqrt{\frac{13-3\sqrt{13}}{2}}$.
28. Simplify $\sqrt{1+\sqrt{21+12\sqrt{3}}}$.

Solve the equations:

29. $\sqrt{x-9}+\sqrt{x-16} = 7$.
30. $\sqrt{5x}-\sqrt{3(2-x)} = 1$.
31. $\sqrt{x^2-1}+\frac{1}{\sqrt{x^2-1}} = 1\frac{0}{3}$.
32. $\sqrt{x-3}+\sqrt{2x+1} = 2\sqrt{x}$.
33. $\sqrt{x+1}+\sqrt{9x+1} = 5$.
34. $\sqrt{2x+37}-2\sqrt{2x+1} = 2\sqrt{10x+14}$.

Examples XXI.

1. Divide $3x^{-\frac{3}{4}}+4x^{-\frac{1}{2}}-13x^{-\frac{1}{4}}+6$ by $x^{-\frac{1}{2}}+2x^{-\frac{1}{4}}-2$.

2. Find the square root of $4x+5+x^{-1}-12x^{\frac{1}{2}}+6x^{-\frac{1}{2}}$.

3. Find the square root of $4x^{\frac{4}{5}}-12x^{\frac{2}{5}}+13-6x^{-\frac{2}{5}}+x^{-\frac{4}{5}}$.

4. If $a^b=b^{-a}$, prove that $\left(\frac{b}{a}\right)^{\frac{b}{a}}=b^{\frac{b}{a}+1}$.

5. Simplify $\sqrt{\left\{\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\right\}}$

6. Find the square root of $(11\sqrt{7}+13\sqrt{5})(\sqrt{7}-\sqrt{5})$

7. Calculate the value of $x+\frac{1}{x}$ when $x=\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

8. Multiply $x^{\frac{2}{3}}+y^{\frac{4}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $x+x^{\frac{1}{3}}y^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{8}{3}}$.

9. Solve
$$2x+3y=5\sqrt{3}+\sqrt{2}.$$
$$(\sqrt{3}+\sqrt{2})x+(\sqrt{3}-\sqrt{2})y=2.$$

10. Give rules for finding the product of two surds. Find the product of $\sqrt{18}$ and $\sqrt[3]{12}$.

11. Find the value of $\frac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}$.

12. Find the sum of $\frac{2+\sqrt{3}}{4+\sqrt{3}}$ and $\frac{1}{5+2\sqrt{3}}$.

13. Calculate, correct to four decimal places, $\frac{3\sqrt{2}+\sqrt{6}}{3+\sqrt{3}}$.

14. Show that $x^m\div x^n=x^{m-n}$ for all values of m and n.

Simplify $\left(\frac{x^4}{y^3}\right)^{-\frac{1}{2}}\times\left(\frac{x^{-\frac{3}{2}}}{x^{-\frac{1}{2}}}\right)^4\div(x^{-1}y^2z^{-3})^3$.

15. Multiply x^a+y^b by $x^{\frac{1}{a}}+y^{\frac{1}{b}}$.

16. Simplify $\sqrt[3]{a^{-\frac{1}{2}}b^{-1}}\times a^{\frac{5}{6}}b^{\frac{1}{2}}\times\sqrt{a^{-1}b^{\frac{3}{2}}}$.

17. State the fundamental laws of indices, and deduce interpretations of $x^{\frac{1}{2}}$ and $x^{-\frac{4}{3}}$.

18. Find the square root of $62+20\sqrt{6}$.

19. Find correct to 3 places of decimals the value of $\frac{4-\sqrt{3}}{4+\sqrt{3}}$.

CHAPTER XXII

LOGARITHMS

198. Ordinary arithmetical numbers are the sums of multiples of powers of ten thus:

$$3746 = 3\times10^3+7\times10^2+4\times10^1+6\times10^0,$$

$$37{\cdot}46 = 3\times10^1+7\times10^0+\frac{4}{10}+\frac{6}{10^2}$$

$$= 3\times10^1+7\times10^0+4\times10^{-1}+6\times10^{-2},$$

$$\cdot03746 = 3\times10^{-2}+7\times10^{-3}+4\times10^{-4}+6\times10^{-5}.$$

The connexion between ordinary numbers and powers of ten is shown very clearly in the following method of expressing numbers, which is often used in scientific work:

$$3746 = 1000\times3{\cdot}746 = 10^3\times3{\cdot}746,$$

$$37{\cdot}46 = 10\times3{\cdot}746 = 10^1\times3{\cdot}746,$$

$$\cdot03746 = \tfrac{1}{100}\times3{\cdot}746 = 10^{-2}\times3{\cdot}746.$$

Examples XXII a.

What is the highest power of ten contained in the following numbers respectively? Express each of them in a form similar to $10^3\times3{\cdot}746$.

1. 1157·3.	2. 243·97.	3. ·0893.
4. ·0008906.	5. 31570·5.	6. 7000·005.

Given that $10^{\cdot9845} = 9{\cdot}651$ and $10^{\cdot5130} = 3{\cdot}258$, express each of the following numbers as a power of 10:

7. 96·51.	8. ·03258.	9. 965·1.
10. ·0009651	11. 32580.	12. ·9651.
13. $9{\cdot}651\times3{\cdot}258$.	14. $32{\cdot}58\div9{\cdot}651$.	15. $(96{\cdot}51)^{10}$.
16. $\sqrt[3]{96{\cdot}51}$.	17. $\sqrt{\cdot03258}$.	18. $3{\cdot}258+9{\cdot}651$.

199. Since $10^x\times10^y = 10^{x+y}$, it is seen that if numbers can be expressed as powers of 10, they can be multiplied by adding the indices of those powers. The other index laws can be used in a similar way. Also the previous section shows that, if the numbers between 1 and 10 can be put in the form 10^y, all other

numbers can be put in the same form by increasing or decreasing the index by a whole number.

An approximate method of finding y so that 10^y shall equal any number x between 1 and 10 is to use the graph of $x = 10^y$. If the student has not already drawn this graph (see Examples XXI b, 24), he should do so now, using as large a scale as possible. This graph is shown on a small scale in Fig. 50.

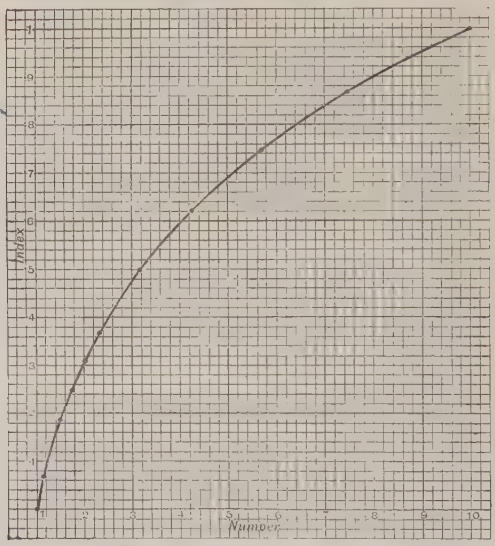


FIG. 50. Graph of $x = 10^y$.

Example. *Find the value of* $\sqrt[3]{\cdot 37} \times 9{\cdot}6^5 \div \sqrt[5]{520}$.

From the graph $3{\cdot}7 = 10^{\cdot 568}$, $9{\cdot}6 = 10^{\cdot 982}$, $5{\cdot}2 = 10^{\cdot 716}$; *

$\therefore \quad \cdot 37 = 10^{-1} \times 3{\cdot}7 = 10^{-1} \times 10^{\cdot 568} = 10^{-1+\cdot 568}$,

$520 = 10^2 \times 5{\cdot}2 = 10^2 \times 10^{\cdot 716} = 10^{2{\cdot}716}$.

$$\begin{aligned}
\text{Hence expression} &= \sqrt[3]{10^{-\cdot 432}} \times (10^{\cdot 982})^5 \div \sqrt[5]{10^{2{\cdot}716}} \\
&= 10^{-\cdot 144} \times 10^{4{\cdot}910} \div 10^{\cdot 543} \\
&= 10^{-\cdot 144+4{\cdot}910-\cdot 543} \\
&= 10^{4{\cdot}910-\cdot 687} \\
&= 10^{4{\cdot}223} \\
&= 10^4 \times 10^{\cdot 223}
\end{aligned}$$

* The third place in the index is estimated by eye and is very doubtful.

From the graph we find that when $y = \cdot 223$, $x = 1 \cdot 7$;

$\therefore$ expression $= 10^4 \times 1 \cdot 7$

$= 17000$ correct to two significant figures.

Examples XXII b.

By means of the graph of $x = 10^y$, find correct to two significant figures the values of:

1. $\sqrt[3]{4 \cdot 7}$.
2. $2 \cdot 7 \times 3 \cdot 9 \times 4 \cdot 6$.
3. $290 \times \sqrt{37}$.
4. $\dfrac{\sqrt{2} \times \sqrt[3]{3}}{\sqrt[4]{4}}$.
5. $(\cdot 079)^3 \div (\cdot 0098)^{\frac{3}{2}}$.
6. $\sqrt[5]{(490)^5}$.

200. **The logarithm of a number to a given base is the index of the power to which the base must be raised to equal the given number**; i. e. if $a^y = x$, y **is the logarithm of** x **to base** a. **This is written** $y = log_a x$.

The common logarithm of a number is the index of the power of ten which equals the number; thus $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, &c.

The logarithm of any number between 1 and 10 is more than 0 but less than 1, i. e. it is a pure decimal. A number greater than 10 can, as shown in § 198, be expressed as the product of a positive integral power of 10 and a number between 1 and 10; a positive number less than 1 can, similarly, be expressed as the product of a negative integral power of 10 and a number between 1 and 10. Hence in general the logarithm of a number consists of a decimal portion (called the mantissa) which is always positive, and a positive or negative integral portion (the characteristic).

It follows from the previous sections of this chapter that—

(i) The mantissa is independent of the position of the decimal point and depends only on the figures that make up the number.

(ii) The characteristic depends only on the position of the decimal point.

(iii) The characteristic is the index of the highest power of 10 that occurs in the number and may be found by

counting from the unit place to the first significant figure,* the unit being counted as nothing; the characteristic is positive if the number is greater than 1, negative if the number is less than one.†

Examples. $\log 3{\cdot}937 = 0{\cdot}5952$
$\log 393{\cdot}7 = 2{\cdot}5952$
$\log {\cdot}03937 = -2 + {\cdot}5952$

The negative sign of a characteristic is usually placed above the digit and the decimal written immediately after the characteristic, so that log ·03937 is written $\bar{2}{\cdot}5952$ and read 'bar 2 point 5952'.

201. The mantissa is found by means of tables. The tables at the end of this chapter give four decimal places, larger tables can be obtained giving seven or more places.

Example. *To find log* 482·3.

Counting from the 2 to the 4, the characteristic is found to be +2.

In the table on p. 294 look for 48 in the extreme left-hand column. In the same line with 48 and under 2 we find 6830, this is the mantissa of log 482; under the 3 in the small columns on the right hand and in a line with 48 we find 3, this must be added to obtain the mantissa of log 4823.

Hence log 482·3 = 2·6833.

202. The number corresponding to a given logarithm may be found from the table of logarithms by reversing the steps of the last section, or may be found from the table of anti-logarithms.

* The first significant figure is the first figure that is not nought.

† Instead of this single rule the following two rules are often given If a number is greater than 1, the characteristic is positive and one less than the number of figures before the decimal point. If a number is less than 1, the characteristic is negative and one more than the number of noughts before the first significant figure.

Example. *To find the number whose logarithm is* 3·7603; *in other words, to find the anti-logarithm of* 3·7603.

Method I. Look in the table of logarithms for the mantissa nearest to 7603.

We find 7597 in a line with 57 and under 5, i.e. ·7597 is log 5·75.

But 7597 is too small by 6, in the right-hand columns we find that above 6 is the number 8; hence ·7603 is log 5·756.

But the characteristic is 3.

$\therefore$ 3·7603 = log $(10^3 \times 5{\cdot}756)$ = log 5756.

Method II. Look in the left-hand column of the table of anti-logarithms for 76, the first two figures of the mantissa, and proceed as in § 201.

antilog ·760 = 5·754, 3 is under 2. $\therefore$ add 2.

$\therefore$ antilog ·7603 = 5·756,

antilog 3·7603 = 5756.

203. Since logarithms are indices, the following formulae can be derived at once from the index laws:

$$\log (ab) = \log a + \log b.$$

Since $a = 10^{\log a}$, $b = 10^{\log b}$; $\therefore$ $ab = 10^{\log a + \log b}$.

$$\log (a \div b) \text{ or } \log \frac{a}{b} = \log a - \log b,$$

$$\log (a^n) = n \log a,$$

$$\log \sqrt[n]{a} = \frac{1}{n} \log a.$$

The proofs of these are left as exercises for the student.

204. A little care is necessary when working with negative characteristics, especially when subtracting or dividing.

Example. *Find the value of* $(\cdot 03752)^{\frac{5}{3}}$.

$$\log (\cdot 03752)^{\frac{5}{3}} = \tfrac{5}{3} \log \cdot 03752$$
$$= \tfrac{5}{3} \times \bar{2}{\cdot}5742$$
$$= \tfrac{1}{3} \times \bar{8}{\cdot}8710 \qquad (-10+2=\bar{8})$$
$$= \bar{3}{\cdot}6237.$$
$$\therefore \quad (\cdot 03752)^{\frac{5}{3}} = \cdot 004205.$$

[The mental work for the division is as follows: The next number above 8 that is divisible by 3 is 9; also $8{\cdot}8 = -9 + 1{\cdot}8$, therefore dividing by 3 we obtain $-3 + \cdot 6$.]

205. In working examples, the student should take care that his equalities are true throughout the work; both sides must be numbers or both sides logarithms.

Example I. *Find the value of* $\dfrac{516{\cdot}5 \times 852}{36500}$.

$$\text{Fraction} = \frac{516{\cdot}5 \times 852}{36500},$$

$$\begin{aligned}\log \text{ of fraction} &= \log 516{\cdot}5 + \log 852 - \log 36500\\ &= 2{\cdot}7130 - 4{\cdot}5623\\ &\quad + 2{\cdot}9304\\ &= 5{\cdot}6434\\ &\quad - 4{\cdot}5623\\ &= 1{\cdot}0811.\end{aligned}$$

$\therefore$ fraction $= 12{\cdot}05$.

Example II. *Find the area of a triangle whose sides, measured in feet, are* 976·5, 394·2, 703·6.

Use the formula Area $= \sqrt{s(s-a)(s-b)(s-c)}$ § 106

$\log$ area $= \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log s - c)\}$

976·5	$s = 1037{\cdot}2$	$\log s = 3{\cdot}0159$
394·2	$s-a = 60{\cdot}7$	$\log s-a = 1{\cdot}7832$
703·6	$s-b = 643{\cdot}0$	$\log s-b = 2{\cdot}8082$
2|2074·3	$s-c = 333{\cdot}6$	$\log(s-c) = 2{\cdot}5232$
1037·2		sum $= 10{\cdot}1305$.

$\therefore$ $\log$ area $= \frac{1}{2} \times 10{\cdot}1305$

$= 5{\cdot}0653$

$= \log 116100$.

$\therefore$ area $= 116100$ sq. ft.

To find log 1037·2. From tables log 1·037 is ·0158, and log 1·038 is ·0162. $\therefore$ log 1·0372 is greater than ·0158 but less than ·0162. For small differences we use the *principle of proportional parts*, viz.

If there are 3 numbers a, b, c whose differences are small, then

$$\frac{\log b - \log a}{\log c - \log a} = \frac{b-a}{c-a} \text{ is approximately true}$$

hence

$$\frac{\log 1{\cdot}0372 - \log 1{\cdot}037}{\log 1{\cdot}038 - \log 1{\cdot}037} = \frac{{\cdot}0002}{{\cdot}001}$$

$\therefore$ $\log 1{\cdot}0372 = \log 1{\cdot}037 + \frac{1}{5} \times {\cdot}0004 = {\cdot}0159$.

Example III. *Find the compound interest on £2135 for 25 years at $2\frac{3}{4}$ per cent.*

[We will first prove the compound interest formula.

£P in 1 year at r% amounts to $£P + P\frac{r}{100} = £P\left(1+\frac{r}{100}\right)$, i.e. the principal at the beginning of the year is multiplied by $1+\frac{r}{100}$ in the course of the year.

But $£P\left(1+\frac{r}{100}\right)$ is principal at the beginning of the 2nd year

$\therefore$ $£P\left(1+\frac{r}{100}\right)^2$ is amount at the end of the 2nd year, and similarly $P\left(1+\frac{r}{100}\right)^n$ is amount at the end of n years.

Using A to denote amount and R for $1+\frac{r}{100}$, we have $A = PR^n$.]

Formula is $A = PR^n$;

$$\begin{aligned}\therefore \quad \log A &= \log P + n \log R \\ &= \log 2135 + 25 \log 1{\cdot}0275 \\ &= 3{\cdot}3294 + 25 \times 0{\cdot}0118 \\ &= 3{\cdot}3294 \\ &\quad + {\cdot}295 \qquad \text{(The 5 is incorrect and we know nothing about the missing fourth figure.)} \\ &= 3{\cdot}6244 \qquad \text{(The 44 is certainly incorrect.)}\end{aligned}$$

$\therefore$ $A = £4211$

and compound interest = £2076.

This answer is correct to two places, to obtain a more correct answer seven-figure logarithms should be used. (See p. 298.)

Example IV. *The volume of a right circular cone is 176·2 c. inches, the radius of the base is 3·2 inches, find the height.*

Formula is $V = \frac{1}{3}\pi r^2 h$,

i.e. $h = \dfrac{3V}{\pi r^2}$.

$$\begin{aligned}\therefore \quad \log h &= \log (3V) - \log \pi - 2 \log r \\ &= \log 528{\cdot}6 - \log 3{\cdot}142 - 2 \log 3{\cdot}2 \\ &= 2{\cdot}7231 - 0{\cdot}4972 - 2 \times 0{\cdot}5051 \\ &= 2{\cdot}2259 \\ &\quad - 1{\cdot}0102 \\ &= 1{\cdot}2157;\end{aligned}$$

$\therefore$ height = 16·44 inches.

Example V. *Find the value of* $2{\cdot}718^{-3{\cdot}142}$.

Expression	$= 2{\cdot}718^{-3{\cdot}142}$	3·142
log expression	$= -3{\cdot}142 \times 0{\cdot}4343$	·4343
	$= -1{\cdot}3646$	1·2568
	$= \bar{2}{\cdot}6354$	943
$\therefore$ expression	$= {\cdot}04319$	126
		9
		1·3646

[We might have found $3{\cdot}142 \times 0{\cdot}4343$ by logs instead of by ordinary multiplication.

Since the tabulated mantissae are all positive we must not leave the final logarithm as $-1{\cdot}3646$, but must actually subtract from 0, thus getting a positive mantissa and negative characteristic.]

Examples XXII c.

Find the values of

1. $\log_2 8$.
2. $\log_8 2$.
3. $\log_2 64$.
4. $\log_9 27$.
5. $\log_2 16$.
6. $\log_2 1{\cdot}6$.

Given that $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$, $\log 7 = {\cdot}84510$, find

7. $\log {\cdot}2$.
8. $\log {\cdot}036$.
9. $\log 14{\cdot}7$.
10. $\log \frac{1}{5}$.
11. $\log \frac{4}{21}$.
12. $\log 12{\cdot}348$.

By using tables find the value of:

13. $30{\cdot}56 \div 4{\cdot}105$.
14. ${\cdot}03056 \times 0{\cdot}4105$.
15. $\sqrt[5]{37{\cdot}24}$.
16. $254{\cdot}3 \div 0{\cdot}09027$.
17. $\sqrt{84{\cdot}05 \times 0{\cdot}1357 \div (1{\cdot}163)^2}$.
18. $(1{\cdot}342 \times 0{\cdot}01731 \div 0{\cdot}0274)^{0{\cdot}317}$.
19. $\log {\cdot}0424 \div \log {\cdot}2165$.
20. $87{\cdot}35 \div (0{\cdot}07568 \times 3{\cdot}501)$.
21. $\frac{4}{3} \times 3{\cdot}142 \times (9{\cdot}67)^3$.
22. $\sqrt{(2{\cdot}63)^4 \div (5{\cdot}834)^{\frac{5}{2}}}$.

Use tables to find the value of:

23. $\sqrt{319{\cdot}2} \times 1{\cdot}756$.
24. $\dfrac{({\cdot}439)^{\frac{2}{3}} \times (14{\cdot}7)^2}{\sqrt[4]{{\cdot}0062}}$.
25. $3{\cdot}142 \times (71{\cdot}43)^{\frac{2}{3}}$.
26. $\sqrt{283{\cdot}6^2 - 245{\cdot}8^2}$.
27. $\dfrac{({\cdot}1136)^{\frac{3}{4}} \times \sqrt{81{\cdot}86}}{\frac{2}{7} \times \sqrt[3]{2000}}$.
28. $(218{\cdot}3)^{-2{\cdot}86}$.
29. $\dfrac{(942{\cdot}6)^3 \times \sqrt{{\cdot}2}}{\sqrt[5]{7{\cdot}892 \times 10^{99}}}$.
30. $(67{\cdot}43)^{1{\cdot}414} - (56{\cdot}29)^{1{\cdot}732}$.

The mensuration formulae for the following examples are given in § 106, p. 140. In all cases the answers should be given as correctly as the data allow.

31. What is the length of the diagonal of a cube whose edge is 7·9 centimetres?

32. Find the volume of a prism whose section is an equilateral triangle of side 1·7 inches and whose length is 8·9 inches.

33. What is the area of a circular racing track 7·32 ft. wide, the outer radius being 792 ft.? Find the side of a square of the same area.

34. What is the diameter of a sphere whose volume is 2000 cubic centimetres?

35. Find the area of the curved surface of a cylinder whose volume is 597 cubic feet and length 8 feet.

36. Find the circumference of the base of a cone whose height is 1·32 ft. and volume 20 c. ft.

37. Find the amount of £215. 10*s.* invested at compound interest for 11 years at $2\frac{1}{2}$ per cent.

38. A sum of £15 is invested in 1908 to accumulate at compound interest at 3 per cent. At the end of what year will it have amounted to £50?

39. What sum will amount to £516 in 16 years at $3\frac{1}{2}$ per cent. compound interest?

40. A man wishes his son to receive £1000 in 15 years' time. If the rate of interest is $3\frac{3}{4}$ per cent., what sum must he invest now?

41. The moment of inertia I of a bar, whose length is a, is given by the formula $I = 2ma^2 \frac{T_1^2}{T_2^2 - T_1^2}$. Find I when $a = 15{\cdot}25$, $m = 907$, $T_1 = 15$, $T_2 = 23\frac{1}{12}$.

42. If $k = \frac{4\pi^2 Il}{T^2}$, find k when $I = 310500$, $l = 201$, $T = 15$.

43. A quantity η is given by the formula $\eta = \frac{8\pi Il}{r^4 T^2}$; given $I = 3{\cdot}105 \times 10^5$, $l = 201$, $r = {\cdot}0555$, $T = 15$, find η.

44. Young's modulus Y is given by the formula

$$Y = \frac{4FL^3}{bd^3l} \times 981;$$

find Y, if $F = 200$, $l = {\cdot}326$, $L = 46{\cdot}5$, $b = 1{\cdot}089$, $d = {\cdot}518$.

45. If $\gamma = \dfrac{\log p_1 - \log p_0}{\log p_1 - \log p_2}$, and if $p_0 = 582{\cdot}4$, $p_1 = 596{\cdot}7$, $p_2 = 585{\cdot}6$, find γ.

46. If $T = 2\pi\sqrt{\dfrac{K}{MH}}$ and if $T = 22$. $K = 52{\cdot}3 \times 8{\cdot}79$, $\log \dfrac{M}{H} = 3{\cdot}1511$, find H.

206. It will be shown in a later chapter that in theoretical work the base 10 is not the most convenient base; it is, therefore, necessary to calculate the logarithms to any other required base when the logarithms to base 10 are known. For this purpose the following theorem is used.

To find the logarithms to base b when the logarithms to base a are known.

Given $\log_a N$ and $\log_a b$, it is required to find $\log_b N$.
Let $\log_b N = x$, then $b^x = N$.
Take the logs of both sides of this equation to base a.

$$x \log_a b = \log_a N$$

$$\therefore \quad x = \frac{\log_a N}{\log_a b}.$$

Example. *Find* $\log_7 15{\cdot}75$.

$$\log_7 15{\cdot}75 = \frac{\log_{10} 15{\cdot}75}{\log_{10} 7} = \frac{1{\cdot}1973}{{\cdot}8451} = 1{\cdot}417.$$

$\log 1{\cdot}1973 = {\cdot}0782,$
$\log \ {\cdot}8451 = {\cdot}9270.$
Diff. $= {\cdot}1512.$

Note.—The formulae of § 203 are true for any base.

207. By means of logarithms equations containing x as an index or part of an index can sometimes be solved.

Example I. *Solve* $3 \,.\, 2^{x+3} = 192 \,.\, 3^{x-3}$.

Rewrite	$3 \,.\, 2^{x+3} = 3 \,.\, 2^6 \,.\, 3^{x-3}$
Divide by 2^6	$2^{x-3} = 3^{x-3}$.
Take logs	$(x-3) \log 2 = (x-3) \log 3$
	$\therefore \ x = 3.$

Example II. *Solve* $3^{3x} = 7^{2x-5}$.

Take logs $\quad 3x \log 3 = (2x-5) \log 7,$

i.e. $\quad 3 \times \cdot 4771 x = (2x-5) \times \cdot 8451$

$\quad 1 \cdot 4313 x = 1 \cdot 6902 x - 4 \cdot 2255.$

Transpose $\quad \cdot 2589 x = 4 \cdot 2255$

$$x = \frac{4 \cdot 2255}{\cdot 2589};$$

and the solution can be finished by ordinary division or by using logarithms.

208. Example. *The following table gives corresponding values of two quantities,* x *and* y *:*

y	10·16	12·26	14·70	20·80	24·54	28·83
x	37·36	31·34	26·43	19·08	16·33	14·04

Try whether x *and* y *are connected by a law* $yx^n = c$*; and if so determine as nearly as you can the values of* n *and* c.

If $\quad yx^n = c,$

then $\quad \log y + n \log x = \log c.$

Denote $\quad \log y$ by v and $\log x$ by $u,$

then $\quad v + nu = \log c.$

If therefore we plot a graph for u and v, i. e. for $\log x$ and $\log y$, the graph will be a straight line.

$\log x$	1·5724	1·4961	1·4221	1·2806	1·2130	1·1473
$\log y$	1·0069	1·0885	1·1673	1·3181	1·3899	1·4599

Fig. 51 shows that the graph is almost exactly a straight line. Therefore we are justified in assuming that $\log y + n \log x = \log c$.

Substitute a pair of values

$$1 \cdot 0069 + n \times 1 \cdot 5724 = \log c,$$

$$1 \cdot 4599 + n \times 1 \cdot 1473 = \log c.$$

Subtract $\quad -\cdot 4530 + n \times \cdot 4251 = 0;$

$$\therefore \quad n = \frac{4530}{4251} = 1 \cdot 06$$

$$= 1 \cdot 1 \text{ to 1 decimal place.}$$

$$\log c = 2 \cdot 7365;$$

$$\therefore \quad c = 550 \text{ to 2 significant figures.}$$

Required law is $yx^{1 \cdot 1} = 550$.

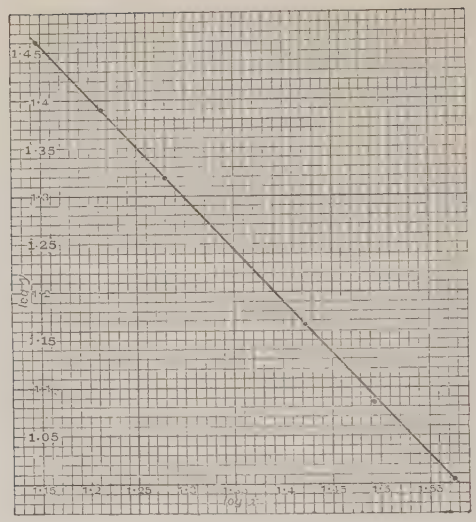


Fig. 51.

Examples XXII.

1. Prove that $\log_a b \times \log_b a = 1$.

2. Find which of the magnitudes $\sqrt[15]{15}$ and $\sqrt[16]{16}$ is the greater.

3. Give reasons for choosing 10 as the base of logarithms. Calculate $\log_2 64$, $\log_2 640$, $\log_2 6{\cdot}4$.

4. Find (i) the number of digits and the first significant figure in 2^{10000}, (ii) the position of the first significant figure in $(\cdot 5)^{10000}$.

5. Prove that $\log_x ab = \log_x a + \log_x b$.

Given $\log 12 = x$, $\log 18 = y$, express $\log 2$ and $\log 3$ in terms of x and y.

6. Why is it that logarithmic tables give only mantissae?

Find the value of $(123{\cdot}4)^{\frac{1}{3}} \times (\cdot 007)^{\frac{1}{3}}$.

Solve the following equations:

7. $3^x = 177147$.

8. $3^{\sqrt{x}} = 243$.

9. $2^{x^2-9x-24} = 4096$.

10. $2^{x+1} + 4^x = 80$.

11. $x^3+y^2=425$, $\log x+\log y=2$.

12. $5^{3x-2y}=3125$, $11^{6x-7y}=14644$.

13. $7^{2x}-6\cdot7^x+5=0$.

14. The following quantities are thought to follow a law like $pv^n=$ constant. Try if they do and find the most probable value of n.

v	1	2	3	4	5
p	205	114	80	63	52

15. At the following draughts in sea-water a particular vessel has the following displacements:

Draught h feet	15	12	9	6·3
Displacement T tons	2098	1512	1018	586

Try to get a simple rule connecting T and h.

16. In some experiments in towing a canal boat the following observations were made; P being the pull in pounds and v the speed of the boat in miles per hour:

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Find an approximate formula connecting P and v.

17. Prove the formula $A=P\left(1+\frac{r}{100}\right)^n$. If A is 130, when P is 100, and n is 7·5, find r.

18. If pv^k is constant, and if $p=1$ when $v=1$, find for what value of v p is 0·2. Do this for the following values of k, 0·8, 0·9, 1·0, 1·1. Tabulate your answers.

19. Calculate to 4 significant figures $2{\cdot}307^{0{\cdot}65}$ and $23{\cdot}07^{-1{\cdot}25}$.

20. $y=a+bx^n$ is the equation to a curve which passes through the three points $x=0$, $y=1{\cdot}24$; $x=2{\cdot}2$, $y=5{\cdot}07$; $x=3{\cdot}5$, $y=12{\cdot}64$; find a, b, n.

21. Find, without assuming the formula,

$$\log_9 741{\cdot}2 \text{ and } \log_4 {\cdot}0563.$$

22. Prove that $a^{\log b}=b^{\log a}$.

	0	1	2	3	4	5	6	7	8	9	1 2 3 4	5	6 7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 9 13 17 4 8 12 16	21 20	26 30 34 38 24 28 32 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 12 15 4 7 11 15	19 19	23 27 31 35 22 26 30 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 11 14 3 7 10 14	18 17	21 25 28 32 20 24 27 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 7 10 13 3 7 10 12	16 16	20 23 26 30 19 22 25 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9 12 3 6 9 12	15 15	18 21 24 28 17 20 23 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 9 11 3 5 8 11	14 14	17 20 23 26 16 19 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8 11 3 5 8 10	14 13	16 19 22 24 15 18 21 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 5 8 10 2 5 7 10	13 12	15 18 20 23 15 17 19 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7 9 2 5 7 9	12 11	14 16 19 21 14 16 18 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7 9 2 4 6 8	11 11	13 16 18 20 13 15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6 8	11	13 15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6 8	10	12 14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6 8	10	12 14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6 7	9	11 13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5 7	9	11 12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5 7	9	10 12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5 7	8	10 11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5 6	8	9 11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5 6	8	9 11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4 6	7	9 10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4 6	7	9 10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4 6	7	8 10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4 5	7	8 9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4 5	6	8 9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4 5	6	8 9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4 5	6	7 9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4 5	6	7 8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3 5	6	7 8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3 5	6	7 8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3 4	5	7 8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3 4	5	6 8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3 4	5	6 7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3 4	5	6 7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3 4	5	6 7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3 4	5	6 7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3 4	5	6 7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3 4	5	6 7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3 4	5	5 6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3 4	4	5 6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3 4	4	5 6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3 3	4	5 6 7 8

These tables are inserted by permission of the Controller of His Majesty's Stationery Office.

The copyright of that portion of the above table which gives the logarithms of numbers from 1000 to 2000 is the property of Messrs. Macmillan and Company, Limited, who, however, have authorized the use of the form in any reprint published for educational purposes.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Logarithms of R for Compound Interest

R	log R	R	log R
1·0025	·00108438	1·0325	·01389006
1·005	·00216606	1·035	·01494035
1·0075	·00324505	1·0375	·01598811
1·01	·00432137	1·04	·01703334
1·0125	·00539503	1·0425	·01807606
1·015	·00646604	1·045	·01911629
1·0175	·00753442	1·0475	·02015403
1·02	·00860017	1·05	·02118930
1·0225	·00966332	1·0525	·02222210
1·025	·01072387	1·055	·02325246
1·0275	·01178183	1·0575	·02428038
1·03	·01283722	1·06	·02530587

Constants used in Mensuration and their Logarithms

	logarithm		logarithm
$\pi = 3{\cdot}14159265$	0·497150	$1 \div \pi = 0{\cdot}31830989$	$\bar{1}{\cdot}502850$
$\frac{1}{2}\pi = 1{\cdot}57079633$	0·196120	$1 \div 4\pi = 0{\cdot}07957747$	$\bar{2}{\cdot}900790$
$\frac{1}{6}\pi = 0{\cdot}52359878$	$\bar{1}{\cdot}718999$	$\sqrt[3]{6 \div \pi} = 1{\cdot}24070098$	0·093667
$\frac{4}{3}\pi = 4{\cdot}18879020$	0·622089	$\sqrt[3]{3 \div 4\pi} = 0{\cdot}62035049$	$\bar{1}{\cdot}792637$
$\sqrt{\pi} = 1{\cdot}77245385$	0·248575	$\sqrt{1 \div \pi} = 0{\cdot}56418958$	$\bar{1}{\cdot}751425$
$\pi^2 = 9{\cdot}86960440$	0·994300	$1 \div \pi^2 = 0{\cdot}10132118$	$\bar{1}{\cdot}005700$
$\sqrt[3]{\pi} = 1{\cdot}46459189$	0·165717	$\sqrt[3]{\cdot^2} = 2{\cdot}14502940$	0·331433
$\pi/180 = 0{\cdot}01745329$	$\bar{2}{\cdot}241877$	$180/\pi = 57{\cdot}29577951$	1·758123

Naperian (or Natural) Logarithms

$e = 2{\cdot}7182182 \quad \log_{10} e = {\cdot}43429448 \quad \log_e 10 = 2{\cdot}30258509$

$\log_{10} N = \log_e N \times \log_{10} e. \qquad \log_e N = \log_{10} N \times \log_e 10$

MISCELLANEOUS EXAMPLES

1. If $x = 1$, $y = -2$, $z = 3$, find the value of

$$\tfrac{1}{2}[x - \tfrac{1}{3}\{y - \tfrac{1}{4}(z - \overline{x - y})\}].$$

2. Show that

$$[(p+q)a-(r-s)b][(p-q)a+(r+s)b]-4qrab \equiv [(p+q)a+(r+s)b][(p-q)a-(r-s)b].$$

3. Find the factors of:

(i) $a(x^2-y^2)+b(x^2-2xy+y^2)$;

(ii) $a^2-4b^2-12b-9$;

(iii) $a^2-b^2-c^2+d^2-2bc-2ad$.

4. From two towns a miles apart, A and B started out at the same time to meet each other, A going p miles an hour and B q miles an hour. How long will it be before A and B meet? How far will they respectively have travelled?

5. Simplify:

(i) $a + \dfrac{x^2+a^2}{2x-a} + \dfrac{x^2+2ax}{a-2x}$;

(ii) $\dfrac{1}{x+2} - \dfrac{1}{x^2-4} - \dfrac{x-8}{x^2+5x+6}$.

6. Prove that $x-a$ is a factor of $f(x)$ if $f(a) = 0$.

Hence, show that $5x^4-2x^3-24x^2+8x+16$ can be divided by x^2-4.

7. Solve:

(i) $\dfrac{1}{x-1} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{x+2}$;

(ii) $2x+y = z+x+3 = y+z$,
$3x+2y-4z = -6$;

(iii) $\dfrac{1}{x-1} + \dfrac{2}{x-2} = \dfrac{4}{3}$.

8. Draw the graph of (i) $y = 2x$, (ii) $x + 2y = 6$.

9. Explain what is meant by the words term, variable, graph, function.

10. A and B have seventeen coins between them, which are all half-crowns or florins. A has twice as many florins as half-crowns and B has two more half-crowns than florins; also A has a shilling more than B. How much has each?

---

11. If $a-b=x=3$ and $a+b+x=2$, find the value of
$$(a-b)\{x^3-2ax^2+a^2x-(a+b)b^2\}.$$

12. Divide $\frac{1}{2}a^3+\frac{3}{2}a^2x-2x^3$ by $\frac{1}{2}a+x$.

13. Simplify
$$\tfrac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\tfrac{2}{3}(x-1)x(x+1).$$

14. Find the factors of:

(i) $ax-2a-3x+6$; (iii) $x^4+4x^3+12x-9$;

(ii) $x^3+x-600$; (iv) $a^3+3a^2b+3ab^2+b^3+c^3$.

15. Find the simplest expression that can be exactly divided by each of the following expressions:
$$a^2-ab,\ b^2+ab,\ a^3-b^3,\ a^3-b^2,\ a^3+b^3.$$

16. If $f(x)=3x^2-5x+4$, find the value of
$$f(x-2)+f(x)-f(x+2).$$

17. Solve the equations:

(i) $a(x+y)-b(x-y)=2a^2$,
$(a^2-b^2)(x-y)=4a^2b$;

(ii) $3x-[2x+4-(3x-5)]=6x+2-[3x-2-(8-2x)]$.

18. A person, having a sum of money to divide among A, B, and C, gave A one-third and £20 more, B one-third of what was left and £20 more, and C one-third of what was then left and £20 more, which exhausted the amount. How much did A, B, and C respectively receive?

19. Show by drawing graphs that the three equations
$$3x-2y=13,\ y=4x-14,\ 2x+5y+4=0$$
are satisfied by the same values of x.

20. Prove that the value of a fraction is unaltered by multiplying its numerator and denominator by the same quantity.

---

21. Prove that $a-(b-c)=a-b+c$, and that

$$a(x+y-z)=ax+ay-az.$$

State any restrictions to the values of the letters that you assume in your proof.

22. Multiply $\frac{1}{4}x^2-\frac{1}{6}x+\frac{1}{9}$ by $\frac{3}{2}x^2+x+\frac{2}{3}$.

23. Find the H. C. F. and L. C. M. of

$$6y^2-y-2,\ 3y^2+5y+2,\ 3y^3-y^2+y+2.$$

24. Find the square root of $\frac{4}{9}x^2+\frac{4}{3}x-1-\frac{3}{x}+\frac{9}{4x^2}$.

25. Apply the ordinary rules for indices to prove that

$$\sqrt[y+z]{\left(\frac{\sqrt[y-z]{x^z}}{\sqrt[y+z]{x}}\right)^{y^2-1}}=x.$$

26. State and prove the rules for 'transposing'.

27. Solve:

(i) $$\frac{1-\frac{1}{x}}{1+\frac{1}{x}}+\frac{1+\frac{1}{x+2}}{1-\frac{1}{x+2}}=2,$$

(ii) $$1-a^4+\frac{a^4-x}{a^4}-\frac{a^3+x}{a^3}+\frac{a^2-x}{a^2}-\frac{a+x}{a}=0.$$

28. State and prove the formula for solving a quadratic equation.

Solve $3x^2+4x-9=0$ by two algebraical methods and one graphical method.

29. A room is such that, if it had been 5 feet longer and 3 feet narrower its area would have been unaltered, and if it had been 5 feet wider, it would have been a square. Find its length and breadth.

30. By drawing a graph find the minimum value of

$$3x^2-5x+13;$$

and verify by an algebraical method.

---

31. If $a=1$, $b=-2$, $c=\frac{1}{2}$, find the value of

$$\frac{a^3}{(a-b)(c-a)}+\frac{b^3}{(b-c)(a-b)}+\frac{c^3}{(c-a)(b-c)}.$$

32. Divide

$$8x^4-18x^3y+9x^2y^2+4xy^3-3y^4 \text{ by } 2x^2-xy-y^2.$$

33. Simplify:

(i) $\dfrac{4}{x-3a}-\dfrac{x-5a}{x^2-5ax+6a^2}-\dfrac{3}{x-2a}$;

(ii) $\dfrac{x^3+a^2}{2a^2+3ax-2x^2}\div\left(\dfrac{2a+x}{2a-x}+\dfrac{a-2x}{a+2x}\right)$.

34. A room a feet long and b feet wide is carpeted in such a way as to leave a margin of c inches wide between the carpet and the walls. Find the area of the margin.

35. If ax^2+bx+c is equal to 0 when $x=1$, equal to 1 when $x=2$, and equal to 6 when $x=3$, find the values of a, b, and c.

36. Solve:

(i) $\dfrac{1}{2x-3}+\dfrac{2}{x-5}=\dfrac{12}{x+4}$;

(ii) $\dfrac{x-4}{3}-\dfrac{y-3}{4}=\dfrac{2x+5}{5}-\dfrac{1-y}{9}=2\frac{1}{3}$.

37. A and B can do a piece of work in 10 days, A and C can do it in 12 days. If the work is done by all three, and B receives £3 more than C, find how much A and B together should receive.

38. Justify the method of solving quadratics by factors.

Draw the graphs of:

(i) $(y-2x+1)(y+2x-1)=0$;

(ii) $(y-2x+1)^2+(y+2x-1)^2=0$.

39. Solve $x^2y-xy^2=6=2xy$.

40. If a man spends 22*s.* a year on tea whatever the price of tea may be, what amounts of tea will he receive when the price per lb. is 12, 16, 18, 20, 24, 28, 33, and 36 pence respectively? Give results to the nearest quarter of a lb. Draw a curve to the scale of 4 lb. to the inch and 10*d.* to the inch to show the number of pounds that he would receive at intermediate prices.

---

41. Find the value of

$$(a+3b+6c)\sqrt{\tfrac{1}{8}+a^2+b^2+c^2}+\frac{4a}{c^2}\left(\frac{a-c}{b}+3abc\right)(3a+2b)^3$$

when $a=1$, $b=-2$, $c=\frac{1}{2}$.

42. Prove the laws (i) $a^m\times a^n=a^{m+n}$; (ii) $(a^m)^n=a^{mn}$.

If these laws are true for all values of m and n, what must be the meaning of $a^{\frac{1}{2}}$?

43. Find the square root of $x^4-8x^2+2x+12+\dfrac{1}{(x+2)^2}$.

44. (i) Find the product of

$$x^2-2x+4,\ x^2+2x+4,\ x-2,\ x+2;$$

(ii) Find the factors of

$$a^2x^2+b^2y^2+1-2ax+2by-2abxy.$$

45. Simplify:

(i) $\dfrac{1+\dfrac{x(x-y)}{1+xy}}{x-\dfrac{x-y}{1+xy}}$;

(ii) $\left(5+\dfrac{126}{x-7}-\dfrac{36}{x-2}\right)\left(5+\dfrac{36}{x+2}-\dfrac{126}{x+7}\right)$.

46. Plot the graph of $y=+\sqrt{x}$. How does it differ from the graph of $x-y^2=0$?

47. Solve:

(i) $\dfrac{x-1}{x+1}-\dfrac{x-2}{x+2}=\dfrac{x-3}{x+3}-\dfrac{x-4}{x+4}$;

(ii) $x^2+4y^2=17,\quad 3x-2y=-1$.

48. If α and β are the roots of $x^2-px+q=0$, prove that $\alpha+\beta=p$, $\alpha\beta=q$. Find the equation whose roots are α^2 and β^2.

49. Divide a straight line 6 inches long into two parts, so that three times the rectangle contained by the two parts shall equal half the square on the whole line.

50. If the value of $4ac-b^2$ is positive, prove that the value of the expression ax^2+bx+c is of the same sign for all values of x.

---

51. What is meant by each of the following words:—factor, term, degree, homogeneous? Write a homogeneous expression of the third degree in x and a. Which, if any, of the following statements are obviously untrue? Give reasons.

(i) $a^3+b^2+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$;

(ii) $(a+b+c)^3=a^3+b^3+c^3+3(bc+ca+ab)+6abc$;

(iii) $(x+a)(x+b)(x+c)$
$=x^3+x^2(a^2+b^2+c^2)+x(bc+ca+ab)+abc.$

52. (i) Resolve x^6+27 into factors;

(ii) Resolve $36a^2+\frac{1}{4}b^2+\frac{1}{9}c^2+\frac{1}{3}bc-4ca-6ab$ into two equal factors.

53. Solve $x^2+3x=6$ correct to two decimal places and verify both solutions.

54. Solve $x+y=2$, $x^3+y^3=3\frac{1}{2}$.

55. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$, prove that $k=\frac{px-qy+rz}{ap-bq+cr}$.

56. A motor car runs from A to B by one road in $37\frac{1}{2}$ minutes. Another car goes by a different road which is 5 miles longer, and travelling 10 miles an hour quicker reaches B in 35 minutes. Find the rates and distances.

57. Draw in a single figure the graphs of

$$y=x-2, \quad y=\frac{(2x-1)(x-1)}{2x+1}.$$

58. A line 6 cm. long is divided into two parts; find, by drawing a graph, where the point of division must be in order that the product of the lengths of the two parts may be as great as possible.

---

59. A glass h inches high, a inches in diameter at the top, and narrowing to a diameter of b inches at the bottom, holds $\frac{h(a^2+ab+b^2)}{132}$ pints. What will a glass hold for which $h=4$, $a=3{\cdot}7$, $b=1{\cdot}7$?

A vessel is to be made to hold a pint, to be 4·5 inches high and to have sides sloping so that the bottom diameter is 2·3 inches less than the top diameter. Find these diameters to the nearest tenth of an inch.

60. Prove that the numerator and denominator of a fraction may both be multiplied or divided by any quantity without changing the value of the fraction.

Reduce to its lowest terms $\frac{8x^3-38x^2+59x-30}{6x^3+13x^2-13x-30}$.

61. Simplify $\dfrac{\dfrac{x}{x+1}+\dfrac{x}{x-1}}{\dfrac{2}{x^2-1}}-\dfrac{4x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$.

62. Solve:

(i) $x^2-(a+b)x+ab=0$;

(ii) $x^2-(a+b)x+ab=c^2$.

Justify the method of solving quadratics by factors.

63. There are two numbers each consisting of the same two digits. Their product is 3478, and one of them exceeds four times the sum of the digits by 3. What are the numbers?

64. Simplify:

(i) $\{(x^2)\sqrt[3]{(y^3)^4}\}^5 \div \sqrt[4]{(x^2)^6(y^4)}$;

(ii) $\sqrt[5]{2^3 \times (2^4)^3 \div 2^5}$.

65. Solve $x+2y=7,$

$$xy-3x+2y-6=0.$$

Verify graphically.

66. Quantities x and y are known to be connected by a relation of the form $y=bx+c$. From the table given below find graphically or otherwise the values of b and c.

x	0	1	2	3	4
y	-1	2	5	8	11

---

67. A square reservoir measuring a yards each way receives the drainage of b acres. It is found that a thunderstorm raised the level of water in the reservoir by c inches. What depth of water fell, if it may be supposed to have fallen uniformly over the whole area, and that all the water which fell found its way into the reservoir? Give a numerical result for the case $a=40$, $b=20$, $c=14$.

68. Simplify $\dfrac{(x+y)^5-x^5-y^5}{x^2+xy+y^2}$.

If $x^3=x+1$, show that $x^5=5x+3$.

69. State the factor theorem in words and symbols. Prove it.

The expression $z^4-8z^3+15z^2+8z-16$ has four factors, find them.

70. Show that $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

By actually solving the equation $x^2-3x+5=0$, show that the product of the roots is 5.

71. Solve $\left.\begin{aligned} 2x+3y &= 7 \\ 3x^2+xy+4y^2 &= 36 \end{aligned}\right\}$.

What kind of curve does the second of these equations represent?

72. A man who walks at the rate of $4\frac{1}{2}$ miles per hour has

20 minutes start of another who walks at the rate of 5 miles per hour. When and where will he be overtaken? Solve graphically.

73. A wine merchant bought a cask of sherry for £9 and, after losing 3 gallons by leakage, sold the rest of the cask at six shillings per gallon above cost price, thereby realizing a profit of $33\frac{1}{3}$ per cent. on his whole outlay. How many gallons did the cask contain?

74. Draw the graphs of

$$y = x^3 - 3x - 4 \quad \text{and} \quad y = \frac{1}{x}.$$

Hence solve the equation $x^3 - 3x^2 - 4x = 1$.

---

75. A goldsmith measures a finger, for which he is to supply a ring, by putting it into a series of graduated holes in a card. The right hole is 18 millimetres in diameter. The ring is to be circular in section, and its thickness (that is the diameter of the circular section) is to be 1·6 mm., so that the outer diameter of the ring is to be 21·2 mm. Assuming that the volume of such a ring of inner diameter p mm. and outer diameter q mm. is $\frac{\pi}{4}(p+q)(q-p)^2$, find the volume of the ring. Find also its weight and value if the metal weighs 16·4 grams per c.c., and is worth 2*s.* 1*d.* per gram.

76. Simplify:

(i) $$\frac{x^2-4xy-21y^2}{x^2-9xy+20y^2} \div \frac{x^2-13xy+42y^2}{x^2+3xy-40y^2};$$

(ii) $$\frac{x+1}{(x-y)(x-z)} + \frac{y+1}{(y-z)(y-x)} + \frac{z+1}{(z-x)(z-y)}.$$

77. Out of £x a man pays a bookseller's bill for £y being allowed 2*d.* in the shilling discount, and also a butcher's bill for £z being allowed 5 per cent. discount. The balance he divides equally among his m children; how much does each get?

78. Solve the equations:

(i) $(2x+3)(x-1) = 4x+6$;

(ii) $$\frac{147}{2+\frac{3}{4x}} = \frac{25}{12}\left(2+\frac{3}{4x}\right);$$

(iii) $x+y+xy = 5, \quad xy-y = 1.$

79. Find by a graphical method the roots of the equation

$$5x^2-14x-81=0.$$

Verify your result by calculation.

80. Define *logarithm, characteristic.* Calculate by logarithms the values of

(i) $\dfrac{293\times601}{359\times773}$; (ii) $401\sqrt{4523}$.

81. A crew which can row at the rate of 10 miles per hour on still water, finds that it takes twice as long to pull up a certain reach against stream as it does to come down the same reach: find how fast the river flows.

82. A rectangular garden is to be divided into six equal rectangular portions by three lines parallel to the sides. If two of the lines are drawn parallel to the length and the remaining one parallel to the breadth of the garden the perimeter of each portion is 224 yards; if two of the lines are drawn parallel to the breadth and the third parallel to the length the perimeter of each portion is 216 yards. Find the area of the garden.

---

83. Simplify

$$(y-z)^2+(z-x)^2+(x-y)^2+2(y-z)(z-x)+2(z-x)(x-y)+2(x-y)(y-z).$$

Prove that the result of adding $\dfrac{x}{x+a}+\dfrac{x}{x+b}+\dfrac{x}{x+c}$ to $\dfrac{a}{x+a}+\dfrac{b}{x+b}+\dfrac{c}{x+c}$ is the same for all values of x.

84. Find values of a and b in order that

$$x^3+(a+1)x^2+(b-3)x-2$$

may be divisible by the square of $x+2$.

85. Find for what values of x the expressions

(i) $\dfrac{3-5x}{8}-\dfrac{2x-1}{7}-5x$, (ii) $\dfrac{2x+5}{3x-7}$

are always positive.

86. State what meanings are attached to $a^{-\frac{1}{2}}$ and $a^{\frac{1}{2}}$, and why.

Prove that $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}=\sqrt{10}$.

Solve $\sqrt{3+\sqrt{x}}+\sqrt{3-\sqrt{x}}=\sqrt{10}$.

87. Solve:

(i) $(x-1)(x-2)+(x-3)(x-4)=2(x-2)(x-3)+x$;

(ii) $\dfrac{1}{x-1}-\dfrac{1}{x+3}=\dfrac{1}{35}$;

(iii) $x^2-y^2=40, \quad x+5=6(y-1)$.

88. A person sells goods at £31. 5*s*. and gains as many pounds per cent. as the goods cost. Find the cost price.

89. The length of a rectangular field is 119 yards and its breadth is 19 yards. How much must the length be diminished, and how much must the breadth be increased, in order that the area of the rectangle may remain the same, while its perimeter is increased by 24 yards?

90. Graph the function $10+3{\cdot}4x-0{\cdot}6x^2$. Find its maximum value and the values of x for which it vanishes.

---

91. Prove that

$$x(a+b+c+d+\ldots)=xa+xb+xc+xd+\ldots.$$

Show that if you multiply together two numbers of m and n digits respectively, the product is a number containing at least $n+m-1$ and at most $n+m$ digits.

92. Show that

$$(a-b)^2(a^4x^6+b^4y^6)+ab(ax^2+by^2)^2\{2(a^2x^2+b^2y^2)-ab(x^2+y^2)\}=(a^2x^2+b^2y^2)^3.$$

93. State and prove the condition that x^2+ax+b may be exactly divisible by $x+c$.

Hence, or otherwise, find the condition that x^2+ax+b and x^2+px+q should have a common factor.

94. If $x^2+3x+4=y$, find what value of y will give equal roots for x. Illustrate your answer graphically.

95. Simplify the expressions:

(i) $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$;

(ii) $\dfrac{(a-b)\sqrt{x-c}}{\sqrt{(x-a)(x-b)}}+\dfrac{(b-c)\sqrt{x-a}}{\sqrt{(x-b)(x-c)}}+\dfrac{(c-a)\sqrt{x-b}}{\sqrt{(x-c)(x-a)}}$.

96. Trace the following graphs (to a scale 1 inch = 1):

(i) $y = 1{\cdot}4x - 0{\cdot}8$;

(ii) $y = 0{\cdot}25x^3$.

Hence solve the equation $x^3 = 5{\cdot}6x - 3{\cdot}2$.

97. The value of the expression $ax + bx^2$ is 3 when $x = 2$, and its value is $-\frac{1}{2}$ when $x = -\frac{1}{2}$; find the values of a and b, and draw a graph showing the values of the expression for all values of x between -5 and $+5$.

98. Calculate, by means of logarithms, the values of

(i) $\dfrac{519^2}{377 \times 803}$; (ii) $\sqrt[5]{23}$.

---

99. Find, by two distinct methods, the value of

$$x^3 - 11{\cdot}8x^2 + 10{\cdot}3x - 121 \text{ when } x = 12.$$

100. In New Zealand a scheme of old-age pensions is in existence, whereby any one of more than 65 years of age is entitled to a pension of £p, provided that his income does not exceed £q. For every pound of income above £q, £1 is taken off the pension, and persons in receipt of income equal to or greater than £$(p+q)$ have no right to a pension. If n be the total number of persons of more than 65 years of age, a the percentage of those with incomes greater than £$(p+q)$ per annum, b the percentage with incomes less than £q per annum, £r the average income of the remainder, show that the annual cost of the scheme to the State is

$$\frac{n}{100}\{(100-a)p + (100-a-b)(q-r)\}.$$

101. Solve:

(i) $2(x-a)(a+b) - (x-3b)(a-b) = 8ab$;

(ii) $(5x-1)(3-x) = 8x$;

(iii) $4x^2 + y^2 = 101, \quad 2x - y = 11$.

102. Simplify:

(i) $\left(x + \dfrac{1}{y}\right)\left(y + \dfrac{1}{x}\right) - \left(x - \dfrac{1}{y}\right)\left(y - \dfrac{1}{x}\right)$;

(ii) $\left(19 - \dfrac{44}{4x-1} - \dfrac{62}{3x+4}\right)\left(23 + \dfrac{22}{3x+2} + \dfrac{124}{4x-5}\right)$;

(iii) $\sqrt{(a^{-\frac{2}{3}} b^3 c^{-\frac{2}{3}}} \div \sqrt[3]{a^{\frac{1}{3}} b^4 c^{-1}}$.

103. Show that if the roots of the quadratic $ax^2+bx+c=0$ are imaginary, then the expression ax^2+bx+c has for all values of x the same sign as a.

104. Draw the graph of $y=x^3$.

Employ it to show that the equation $x^3-3x+1=0$ has three real roots, and determine, roughly, their values.

105. Prove that $\log a^p = p \log a$ and $\log a - \log b = \log \frac{a}{b}$.

If $\log 2 = x$, $\log 3 = y$, find in terms of x and y the logarithms of $\frac{2}{9}$, 75, ·0015.

106. The law connecting the pressure and volume of a given mass of gas is $pv^\gamma = C$ where p is the pressure, v the volume, and γ and C are constants. If the gas is such that its pressure is doubled when its volume is compressed from 100 c.c. to 61 c.c., find the value of γ.

---

*107. Un triangle a pour côtés $a=13$, $b=14$, $c=15$; calculer :

(i) la surface S ;

(ii) le rayon R du cercle circonscrit donné par la formule

$$R = \frac{abc}{4S}.$$

108. De $\frac{1}{2}a^2x+\frac{1}{4}ax^2+\frac{2}{3}x^3$ retrancher $-\frac{3}{4}a^2x+\frac{1}{3}ax^2+\frac{5}{8}x^3$.

109. Déterminer m et n de telle sorte que le polynôme

$$x^4-3x^3+mx+n$$

soit divisible par x^2-2x+4.

110. Extraire la racine carrée des polynômes suivants :

(i) $a^{2m}+2a^m y^n+y^{2n}$; (ii) $1-2x+5x^2-4x^3+4x^4$.

111. Calculer les puissances et les racines suivantes :

(i) $(\sqrt[3]{\sqrt{8a^3}})^7$; (ii) $\left[\frac{c^3d}{(a+b)^{\frac{3}{2}}}\right]^{-\frac{1}{3}}$.

112. Résoudre les systèmes d'équations suivantes :

(i) $x+y=16$, $y+z=22$, $z+x=38$;

(ii) $$\frac{1}{3x-2y+1}+\frac{1}{x+2y-3}=\frac{5}{12},$$
$$\frac{1}{x+2y-3}-\frac{1}{3x-2y+1}=\frac{1}{12}.$$

* In some London University Examinations a knowledge of French and German mathematical terms is required. A few questions in these languages may, therefore, be found useful.

113. Construire la courbe représentée par l'équation

$$y = \frac{1}{1+x^2}.$$

114. Démontrer les propriétés suivantes :

(i) La caractéristique du logarithme d'un nombre plus grand que 1 contient autant d'unités moins un que le nombre a de chiffres à sa partie entière ;

(ii) Le logarithme d'une racine d'un nombre égale le logarithme de ce nombre divisé par l'indice de la racine.

---

115. Verwandle in ein Produkt von zwei oder mehr Faktoren :

(i) a^5+b^5 ; (ii) $8a^3+12a^2+6a+1$.

116. Löse die Klammern und vereinfache möglichst :

(i) $(c-b)(x+a)^2+(a-c)(x+b)^2+(b-a)(x+c)^2$;

(ii) $\left(\frac{x}{a}+\frac{a}{x}\right)\left(\frac{y}{b}+\frac{b}{y}\right)-\left(\frac{x}{a}-\frac{a}{x}\right)\left(\frac{y}{b}-\frac{b}{y}\right)$.

117. Löse die folgenden Gleichungen :

(i) $(4x-\frac{1}{2})(5x+\frac{3}{4}) = (2x+\frac{1}{2})(10x-\frac{3}{2})$;

(ii) $(x+1)^3 = x^3+1+3x(x+1)$;

(iii) $0{,}5x-y = \frac{1}{2}$; $0{,}6x+y = \frac{3}{5}$.

118. Finde eine Ungleichung von der Form $x<a$ oder $x>a$, wo a eine bekannte Zahl ist :

(i) $7x-3>4(x-1)$;

(ii) $\frac{1-4x}{12}-\frac{2+x}{9}+\frac{4-3x}{8}-\frac{5}{6}(1+x)>x$.

119. Verwandle in eine Potenz mit gebrochenem Exponenten : $\sqrt[4]{b^7}$, $\sqrt[4]{a^{4+b}}$. Vereinfache $(a^{\frac{1}{3}} \div a^{\frac{3}{10}})^{-5}$.

Berechne logarithmisch auf zwei Dezimalstellen

$$\sqrt[5]{\frac{13{,}43\times 5{,}67}{(0{,}912)^{\frac{1}{2}}}}.$$

120. Mit wieviel Ziffern wird $5^{\binom{5}{5}}$ geschrieben ?

Löse $7^{3x+2}+4^{x+2} = 7^{3x+1}+4^{x+3}$.

EXAMINATION PAPERS

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

Examination for Naval Cadetships. July, 1906.

1–10. Arithmetic.

11. Find the value of $2x^3-x^2-x-2$ when $x=1{\cdot}5$.

12. Resolve into factors:

$$4x^2-20x+25,\quad 2x^2-x-15,\quad (3x-2)^2-(2x-3)^2.$$

13. Simplify (i) $2(x+y)^2-(4x-3y)y-2(x^2-y^2)$;

(ii) $\dfrac{a^2-a^3}{a-a^3}\times\dfrac{a^2+a^3}{a+a^3}$.

14. Solve the equations:

(i) $\dfrac{2x-3}{5}-\dfrac{3x-2}{11}+\dfrac{1}{10}=0$;

(ii) $3x+y=2,$
$x-3y=9.$

15. A certain sum exceeds £5 by as much as £3 exceeds the half of that sum. What is the sum?

Examination for Naval Cadetships. July, 1907.

1–5. Arithmetic.

6. Multiply $1-2a+3a^2$ by $3-2a+a^2$.
Divide $6+a-16a^3$ by $3-4a$.

7. If $x=-2$, find the value of

$$\frac{1-x}{3+x}-\frac{1-x^2}{5-x^2}-\frac{2+x}{x^2}.$$

8. Resolve into simple factors:

$$6a^2+13a+6,\quad 6a^2+5a-6.$$

9. Simplify $\dfrac{3b-2a}{ab}+\dfrac{4c-2b}{bc}-\dfrac{3c-2a}{ca}$.

10. Solve the equation

$$\frac{2x-5}{3}+\frac{3x-7}{2}-\frac{4x-9}{4}=0.$$

11. Solve the simultaneous equations:

$$\frac{x}{2}-\frac{y}{5}=1,$$

$$\frac{8x}{3}-\frac{3y}{2}=1.$$

12. A farmer sells 6 more than one-third of his flock of sheep; then he sells 7 more than one-third of the remainder. He has still one-third of his original flock. What was the number of it?

ARMY QUALIFYING EXAMINATION.

September, 1907.

[Candidates must not attempt more than six questions.]

1. Multiply $3\sqrt{xy^3}-xy+2\sqrt{\dfrac{y^5}{x}}$ by $\sqrt{x^3y}-2\sqrt{xy^3}$.

2. Divide $8a^3-b^3+c^3+6abc$ by $c+2a-b$.

3. Prove that the expressions

$$2x^3-3x^2-2x+3,\quad 4x^3-8x^2+x+3,\quad 4x^3-4x^2-5x+3$$

have a common factor.

4. Solve the equations:

$$6x+4y-2z-5=0,$$
$$3x-2y+4z+10=0,$$
$$5x-2y+6z+13=0.$$

5. Simplify the expression

$$\frac{(x+y-z)^2-w^2}{(x+y)^2-(z+w)^2}+\frac{(y+z-x)^2-w^2}{(y+z)^2-(x+w)^2},$$

and find the value of the simplified expression when

$$x=3,\quad y=-1,\quad z=4,\quad w=-6.$$

6. Solve the equation $2\sqrt{x}-\sqrt{5x+1}+\frac{1}{2}=0$.

7. Draw the graphs of the equations

$$4x+2y-17=0, \quad y=-2x^2+8x-2;$$

name the values of x and y common to both, and show clearly the directions which you take as positive along the axes of reference.

8. A man pays a bill for 5 tons of coal and 22 thousand cubic feet of gas, the whole amounting to £10. 13*s*. If the cost of coal were 8 per cent. lower per ton and that of gas $12\frac{1}{2}$ per cent. lower per thousand the bill would have been £9. 12*s*. Find the prices of coal per ton and gas per thousand.

OXFORD LOCAL EXAMINATIONS.

PRELIMINARY. 1907.

ELEMENTARY.

1. Find the quotient and remainder when

$$x^4-3x^3+x^2+4x+2 \text{ is divided by } x^2-x+1.$$

2. Find the value of

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)},$$

when $x=2,\ y=1,\ z=-3$.

3. Find the H. C. F. and L. C. M. of

$$x^3-x^2-x+1, \quad x^3-3x+2.$$

4. Simplify $\dfrac{2}{a-3b}-\dfrac{1}{a-2b}$, and divide the result by $a-b$.

5. Solve the equations:

(i) $(x+2)(x+3)=(x+4)(x+5)+2$;

(ii) $\dfrac{1}{x}+\dfrac{2}{x-1}=\dfrac{3}{x-2}$.

6. A man buys oranges at the rate of p a shilling and by selling them at q pence a dozen makes a profit of r per cent. Show that $25pq-36r=3600$.

7. A man has been saving annually a quarter of his income. His income is increased by £50 and now he saves a third of his income and finds that he is saving annually £25 more than before. What was his original income?

Higher.

1. Divide $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$ by $a+b+c$.

2. Resolve into factors:

   (i) $4x^2+15xy-4y^2$; (ii) $a^2-2ab+b^2-a+b$.

3. If $2b=a+c$, show that $(a-b)^2+2b^2+(b-c)^2=a^2+c^2$.

4. Solve the equation $\dfrac{x+2}{x+3}+\dfrac{x+3}{x+4}=\dfrac{1}{2}$.

5. (i) Divide $x^{\frac{3}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}-2y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$

   (ii) Simplify $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}+\dfrac{3-\sqrt{7}}{3+\sqrt{7}}$.

6. A man has £200 invested, partly at 4 per cent. and partly at $3\frac{1}{2}$ per cent., and his total income from these investments is £7. 12*s*. How much is invested at 4 per cent. and how much at $3\frac{1}{2}$ per cent.?

CAMBRIDGE LOCAL EXAMINATION.

Preliminary. 1907.

[Candidates can pass in Algebra by doing sufficiently well in Part I of this paper. Processes not algebraical are permitted.]

Part I.

1. Simplify $x-[y-2\{x-(y+x)+2y\}+2x]$.

2. Add the product $(a+b+1)(a-b+2)$ to the product

$$(a+b-1)(a-b-2);$$

and verify the result when $a=4$, $b=1$.

3. Divide $x^4+x^3-5x^2+13x-6$ by x^2+3x-2.

4. Simplify:

(i) $$\frac{\frac{1}{a}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\left(\frac{1}{a}+\frac{1}{b}\right)}{\left(\frac{1}{a}+\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2};$$

(ii) $$\frac{x-1}{x+2}\left\{1-\frac{1}{x+1}-\frac{2}{x^2-1}\right\}.$$

5. Solve the following equations, verifying your solutions:

(i) $$\frac{x-5}{5}=\frac{x-4}{10}+\frac{x-3}{40};$$

(ii) $x+2y=3a-b$, $2x+y=3a+b$.

6. Divide 100 into 3 parts, such that $\frac{1}{4}$ of the first, $\frac{2}{5}$ of the second, and $\frac{3}{5}$ of the third are all equal.

Part II.

7. Resolve into factors:

(i) $a^2+ab+ac+bc$; (ii) $x^3-6x-187$;

(iii) $(x-2)^3-(x-2)$.

8. Find the H. C. F. and the L. C. M. of

$$x^3-3x-2 \quad \text{and} \quad x^3-x^2-5x-3,$$

expressing them as products of factors.

9. Solve the equations:

(i) $x^3-7x=30$;

(ii) $$\frac{x-1}{x+1}+\frac{x-2}{x+2}=1$$

correct to 2 decimal places.

10. A question on progressions.

11. Taking one inch as unit, draw with the same axes the graph of the lines $2x+y=2$, $2x+6=3y$, $y=1$; and find from your figure the co-ordinates of the three points at which the lines intersect.

COLLEGE OF PRECEPTORS.

Third Class. Christmas, 1907.

1. Rearrange the following expression in ascending powers of x, and show, by means of brackets, what are the coefficients of the different powers:

$$3bx^2-7ax-ax^3-6b-2bx^2+bx^3-1-7ax^2.$$

Find also the value of the expression when $a=2$, $b=0$, $x=-1$.

2. Multiply $7p^5-2p^4q-6p^3q^2$ by $3p^2-pq+q^2$.

3. Divide $x^6-2x^3y^3+y^6$ by x^3+xy+y^2.

4. Without more working than is necessary, calculate the coefficient of x^3 in the product of

$$x^4-29x^3-20x^2+5 \text{ by } 2x^3-x^2-x+1.$$

5. State which of the following expressions cannot be resolved into factors, and write down the factors of such as can be so resolved:

$$2a+2b,\quad a^2-b^2,\quad a^2+b^2,\quad a^3-b^3,\quad a^3+b^3,\quad a^4-b^4,\quad a^4+b^4.$$

6. Find the H. C. F. of $2x^2-9x+7$ and $3x^2-5x+2$;

and the L. C. M. of $(x+1)^2$, $(x-1)^2$, x^2-1.

7. Simplify the expressions:

(i) $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{a-b}{ab}$;

(ii) $\dfrac{a}{x-y}+\dfrac{b}{x+y}-\dfrac{ax-by}{x^2-y^2}$;

(iii) $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\div\left(\dfrac{y}{x-y}-\dfrac{x}{y-x}\right)$.

8. Solve the equations:

(i) $\frac{2}{3}x-5=\frac{5}{12}x+\frac{1}{4}$;

(ii) $a^2x-c=b^2x+d$;

(iii) $\dfrac{3x-1}{7}+\dfrac{6-x}{4}-\dfrac{x-2}{6}=2-\dfrac{x+2}{28}$.

9. If two men start together in the same direction, walking b and c miles per hour respectively, c being greater than b, find the distance between them after d hours.

Also, if they start at a minutes after noon, how many minutes will the faster man be in walking x miles, and by what o'clock will he accomplish that distance?

10. Boys equal in number to one-fifth of a form are promoted out of it and twice as many promoted into it; also four new boys join the form. It is then found that the number of boys in the form has been increased by one-third. Find the numbers in the old and new forms.

Second Class or Junior. Christmas, 1907.

1. Write down the following products:

(i) $(x-2y)(2x-3y)$; (ii) $(3x+y)(3x-4y)$

(iii) $(2x+5y)(x+y)$; (iv) $(a+b)(a-b+c)$.

2. Find the square root of

$$4x^4+8xy^3-4x^2y-15x^2y^2+16y^4.$$

3. Find the H. C. F. of

$$1-x^2+2x^3-x^4 \quad \text{and} \quad 1-3x+4x^3-3x^5+x^6.$$

4. Resolve into factors:

(i) $x^2+20x-96$; (ii) $x^2-20x+96$;

(iii) $2x^2y-8y^3$; (iv) $x^2+2xy+y^2-x-y$.

5. Solve the equations:

(i) $\frac{2}{3}(x-6)-\frac{2x-13}{5}=\frac{x}{3}-3$;

(ii) $\frac{2x-y}{7}-\frac{2y-x}{5}=0, \quad 4x-5y=-9$;

(iii) $2x+y=15, \quad 3y-z=13, \quad x+y-5z=-87$.

6. Simplify:

(i) $\frac{-x+5}{2x^2-5x+2}-\frac{x-5}{2x^2+x-1}+\frac{x-7}{x^2-2x-3}$;

(ii) $\frac{a^2-b^2-c^2+2bc}{a^2-b^2+c^2+2ac}$.

7. (i) A rectangular lawn x feet wide and y feet broad is surrounded by a path z feet wide. What is the area of the path?

(ii) Apples cost x shillings a score. How many will be obtained for a sovereign? How many more will be obtained if the price is lowered a shilling a score?

(iii) A cistern can be filled by one tap in x minutes, by another in y minutes. If both are opened, in what time will it be filled?

8. Solve the equation $\frac{1}{x-2} - \frac{1}{x-3} = \frac{-1}{4x-2}$.

The numerator of a fraction is greater than the denominator by 7; if the numerator and denominator are each increased by 18, the two fractions differ by 2. Find the fraction.

9. A train starts from a station X at midday and arrives at Y at 4 p.m. Another train starts from Y at 1 p.m. and arrives at X at 3.30 p.m. They meet 28 miles from X. Find the distance from X to Y.

Either,

10 A. A man can make a articles of one kind in a week. He can make b articles of another kind in a week. How many can he make in a week if he make the same number of each?

Or,

10 B. Draw a graph of $4-x-x^2$, and find from it the greatest value of the expression.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

EXAMINATION FOR SCHOOL CERTIFICATES. JULY, 1906.

ALGEBRA.

[In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in Part I.]

PART I.

1. Add together $(a-b)^4$ and $4a(a+b)^3$, and subtract from your result $a^2(5a+3b)(a+b)$.

2. Find the H. C. F. of x^3+3x^2+16 and x^5-5x^2-48, and the L. C. M. of $18(x^3-6x^2+11x-6)$ and $12(x^2-7x+12)$.

3. Solve the equations:

(i) $\dfrac{1}{x-1}-\dfrac{2}{x-2}+\dfrac{1}{x-3}+\dfrac{x}{(x-1)(x-2)(x-3)}=0$;

(ii) $x^2+28x=165$.

4. Prove that, if $x+y=1$,

$$x^3(y+1)-y^3(x+1)-x+y=0.$$

5. Simplify the expressions:

(i) $\dfrac{\dfrac{1-x}{x}+\dfrac{4}{1-x}}{\dfrac{1+x^2}{x^2}-\dfrac{4}{1+x^2}}$;

(ii) $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-c)(b-a)}-\dfrac{a+b}{(c-a)(c-b)}$.

6. Round the edge of a rectangular court which is half as long again as it is broad runs a path 3 feet wide. If the area of the path is 1584 square feet, find the dimensions of the court.

[The following questions are to be attempted only by Candidates who do not attempt Part II.]

A. Resolve into factors:

(i) x^4-10x^2+9;

(ii) $a^3-a^2b+ab^2-b^3$.

B. Solve the equations:

$$x+y=2a+b,$$
$$3x+2y=7a+3b.$$

C. Prove that, if a, b, c are three consecutive whole numbers,

$$a^3+c^3=2b(b^2+3).$$

D. Draw the graph of the function $3x^2-10x+1$ from $x=-1$ to $x=4$, and determine from your figure the values of x for which the value of the function is -4.

Higher Certificate. July, 1906.

[In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in Part I.]

Part I.

1. Find the value when $a=2$, $b=-1$, $c=-2$, of:

(i) $3c-2ac^2+\dfrac{4c^4}{ab}$;

(ii) $\dfrac{(b^2+1)^2-2ab+1-a^2-b^2}{a+b+b^2+1}$.

2. Multiply $a^2+9b^2+c^2+3bc+ac-3ab$ by $a+3b-c$.
Divide $3a^4-5a^3b+6a^2b^2-3ab^3+b^4$ by a^2-ab+b^2.

3. Simplify:

(i) $4(4x+1)^2-6(4x-1)(x-1)-4(x+1)(10x-1)$;

(ii) $\dfrac{bc(a+b)}{(a-b)(a-c)}+\dfrac{2abc}{(b-c)(b-a)}+\dfrac{ab(b+c)}{(c-a)(c-b)}$

Find the H. C. F. of $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$

4. Draw the graphs of $x=1+y$ and $2x+4y=17$ in the same diagram. Hence solve the two equations simultaneously. [Take 1 inch as your unit.]

5. Solve the equations:

(i) $(x+5)(x-3)=(x-5)(x+3)+4$;

(ii) $ax-by=2(a^2-b^2)$, $x-2y=-3b$;

(iii) $3x^2+x-2=0$.

6. A man buys a certain number of photographs for £1. Two get damaged, and by selling the remainder for 2*d.* each more than they cost he makes one shilling profit. How many did he buy?

[The following questions are to be attempted only by Candidates for Certificates who do not attempt Part II.]

A. Factorize:

(i) x^3-y^3; (ii) $ab(x^2-y^2)+xy(a^2-b^2)$.

B. Find the H.C.F. of x^3+x^2-2 and x^3+2x^2-3.

Show that $(x+y+z)^3-x^3-y^3-z^3=3(y+z)(z+x)(x+y)$.

C. Solve:

(i) $(a^2+x)(b^2+x)=(ab+x)^2$;

(ii) $a(x^2+1)=x(a^2+1)$.

D. A is 8 years older than B; 10 years ago he was twice his age. Find their present ages.

CENTRAL WELSH BOARD.

Elementary Mathematics, Junior Certificate.

July, 1906.

1. Find the value, when $a=-\frac{1}{3}$, $b=2$, $c=-\frac{1}{2}$, of

$$\frac{2a}{(2a+b)(2a+3c)}-\frac{b}{(b+2a)(b-3c)}-\frac{3c}{(3c+2a)(3c-b)}.$$

2. Simplify:

(i) $(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)$;

(ii) $\left(\frac{2}{x-3}-\frac{1}{x-2}\right)\left(\frac{1}{x-1}+\frac{1}{x-3}\right)$.

3 Resolve into factors the following expressions:

(i) $(2x-a)^2-(x-3a)^2$;

(ii) $x^3-x-2(x^2-1)$;

(iii) $p^2-2pq+q^2-3p+3q$.

4. Simplify $\frac{1}{x-2}-\frac{1}{x-1}-\frac{1}{(x-1)^2}$.

5. Solve the equations:

(i) $3(x-1)+5(x-2)+\frac{7x+8}{3}=0$;

(ii) $\frac{x-a}{b}+\frac{x-b}{a}=2$.

6. Solve the equations $3x-4y=1$, $2y-x=1$ both graphically and algebraically.

7. A man saves annually one-third of his income. When his income is increased by £100 a year, he only saves one-fourth, but finds that he is saving annually £12. 10*s.* more than before. What was his original income?

8. A man buys two horses for £70. By selling one of them at a profit of 10 per cent. and the other at a profit of 20 per cent., he gains altogether £10. What did he pay for each?

Elementary Mathematics, Junior Certificate.

July, 1907.

1. Multiply together $a^3-3ab^2+2b^3$ and $a^3-3a^2b+4b^3$, and divide the product by $a^4-5a^2b^2+4b^4$.

2. Resolve into factors:

(i) $2x^2-3xy-2y^2$;

(ii) $(x-2)^3-(x-2)$;

(iii) $a(a+1)-b(b+1)$.

3. Find the H. C. F. of x^4+x^2+1 and $x^4+2x^3+x^2-1$; and find the L. C. M. of these two expressions and

$$x^5-x^4-x^3+3x^2-3x+1.$$

4. Simplify:

(i) $\dfrac{1}{a-1}-\dfrac{1}{a+1}+\dfrac{2}{a^2+1}$;

(ii) $\dfrac{b(a+b)+c(a+c)}{(a+b+c)^2-(a-b+c)(a+b-c)}$.

5. Solve the equations:

(i) $\frac{2}{3}(x-6)+\frac{3}{4}(x-2)+\frac{1}{6}(x-1)=4x-17$;

(ii) $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x}$.

In the case of the second equation, verify your solution.

6. Show, by drawing graphs, that the values of x and y which satisfy the equations $x+2y=1$, $3x-y=2$, satisfy also the equation $2x-3y=1$.

7. A man starts to bicycle from A to B at the rate of 12 miles an hour. When he has completed half the journey his bicycle breaks down and he starts to walk, doing the next 2 miles in half an hour. He is now picked up by a conveyance, which takes him the rest of the way at the rate of 8 miles an hour. If the whole journey occupies an hour and a half, what is the distance from A to B?

8. A grocer has two kinds of tea. When mixed in the proportion of 5 : 1 the mixture is worth 1*s.* 11*d.* a pound; when mixed in the proportion of 1 : 2 the mixture is worth 1*s.* 8*d* a pound. What is the value of a pound of each kind of tea?

Practical Mathematics. July, 1906.

[1. Explain the nature and use of the slide rule.]

2. Determine the factors of:

(i) a^3-b^3; (ii) $3x^2-13x-10$;

(iii) $a^2-2ab+b^2-ac+bc$.

3. Prove that $\log ab = \log a + \log b$.

Take out of the tables the logarithms of 327 and ·00327.

Use tables to find approximately the value of $50^{\frac{1}{5}}$, $(\frac{2}{7})^5$.

4. Draw the graphs of $2x+1$ and x^2 between the values ± 4 of x.

Use the graphs to find, correct to the first place of decimals, the roots of $x^2-2x-1=0$.

5. At noon on Monday one clock, which loses 10 minutes a day, is an hour fast, and a second clock, which gains a quarter of an hour a day, is 20 minutes slow. When will they both indicate the same time?

Draw graphs to represent the variation of each clock, and by means of the graphs verify your solution.

[There were also questions in Arithmetic and Geometry.]

Additional Mathematics. July, 1907.

1. If y is a function of x, and corresponding to the values of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 of x the observed values of y are 10, 9·9, 9·8, 9·5, 9·1, 8·7, 8, 7·1, 6, 4·4, 0 respectively, represent y graphically for values of x between 0 and 10, and find the values of y when $x =$ (i) 4·5, (ii) 8·5.

2. Find the condition that the roots of the quadratic equation $ax^2 + bx + c = 0$ should be real.

If α and β are the roots of the equation

$$x^2 - 3x + 1 = 0,$$

find the equation whose roots are $\dfrac{\alpha}{1+\beta}$, $\dfrac{\beta}{1+\alpha}$.

3. Solve the equations:

(i) $xy = 1$, $x^{\frac{2}{3}} - y^{\frac{2}{3}} = 1\frac{2}{3}$;

(ii) $\sqrt{4-x} + \sqrt{6+x} = \sqrt{6(x+1)}$.

4. (i) Find the square root of $5 + \sqrt{21}$.

(ii) Simplify $\dfrac{\sqrt{5}+2}{\sqrt{5}-2} \div \dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

5. Interpret, justifying your interpretation, a^0, $a^{-\frac{p}{q}}$.

Simplify $(a^{\frac{1}{2}}b^{\frac{3}{2}})^{-2} \times (b^{-\frac{3}{2}}c^{-\frac{1}{3}})^{-3} \div (a^2 b^{-5} c^2)^{-\frac{1}{2}}$.

6. Prove that $\log \dfrac{a}{b} = \log a - \log b$.

If $\log_{10} 2 = x$, $\log_{10} 3 = y$, $\log_{10} 7 = z$, express in terms of x, y, and z the logarithms to the base 10 of 125, ·147, $\sqrt[3]{\cdot 002205}$.

[7, 8, and 9 were beyond the scope of this book.]

INDEX

Abscissa, 76.
Addition, sign of, 9.
  of algebraic expressions, 16, 40.
  of negative numbers, 39.
  of fractions, 183.
Association law, 132.
Asymptote, 224.
Axioms, 84.
Axis, 66.

Base of a logarithm, 200.
Binomial, 130.
Brackets, 18, 54, 135.

Characteristic, 200.
Circle, equation of, 228.
Coefficient, 13.
Common factor, 176.
  multiple, 176.
Commutation laws, 133, 134.
Conditional equation, 84.
Constant, 63.
Co-ordinates, 76.
Cube, 29.
  of a sum, 123.
Cubes, difference of, 170.

Degree of an expression, or equation, 130.
Detached coefficients, 155.
Difference of squares, 164
Dimensions, 130.
Distribution law, 135 *n*.
Division, signs of, 14.
  with negative quantities, 48.
  with algebraical expressions, 111.
  with powers, 111.
  mental, 113, 154.
  with fractions, 186.

Eliminate, 91, 246.
Ellipse, equation of, 229.
Equations, conditional, 84.
  identical, 84.
Equations, simple, 84.
  literal, 87.
  simultaneous, 91.
  of a graph or curve, 139.
  simultaneous quadratic, 166.
  fractional, 191.
  quadratic, 201.
  indeterminate, 256.
  involving surds, 278.
Evolution, 121.
  law, 265.
Exponent, 121.

Factor, 13.
  theorem, 152.
Factors, 162.
Formulae, 78, 139.
Fractions, 180.
Function, 64.
Functional notation, 137.
Fundamental laws, 131.

Graph, 67.
  of statistics, 67, 138.
  of a function, 70, 139.
  of simple equation, 94.
  of quadratic equation, 205.
Graphical illustrations, 25, 43, 54.
  solution of quadratics, 205.
    of simultaneous equations, 94, 248.

Homogeneous, 130, 245.
Horner's method of division, 116, 156.
Hyperbola, 223, 230.

Identity, 84, 150.
Imaginary roots, 122, 202.
Indeterminate equation, 256.
Index, 29.
  laws, 264.
  fractional or negative, 267.
Infinity, 223.
Integral expression, 181.

Interpolation, 189.
Involution, 120.
  law, 265.
Irrational, 191.

Like terms, 16.
  signs, 48.
Linear equation, 130.
Logarithm, 283.

Mantissa, 200.
Maximum, 233, 257.
Mensuration formulae, 140.
Minimum, 233, 257.
Monomial, 130.
Multiplication, signs of, 13.
  with negative quantities, 47.
  of algebraic expressions, 102.
  of powers, 103, 264.
  mental, 104, 154, 162.
  of fractions, 183.
  of roots, 273.

Negative quantity, 32.
  addition, &c., 38.

Ordinate, 76.
Origin, 45.

Plotting, 66.
Positive number, 34.
Power, 29.
Problems, 78, 144, 251.
  graphical, 241.
Product, 13.

Quadratic expression, 130.
  surd, 196.
  equations, 201.
  graphical solution, 205.
Quotient, 111.

Radical sign, 121.
Rational, 174.
Rationalizing, 275.
Rectangle, 27.
Remainder, theorem, 151.
Root, of an equation, 84.
  of a quantity, 121.

Signs of equality and inequality, 130.
Simultaneous equations, 91.
  graphical solution, 94, 205.
  quadratics, 242.
Solution of an equation, 85.
Square, 29.
  of a sum, 122, 125.
Square root, 126.
  of a quadratic surd, 277.
Subtraction, sign of, 9.
  of negative numbers, 41.
  of algebraical expressions, 43.
  of fractions, 183.
Sum, algebraic, 17.
Surd, 272.
Symmetry of curves, 219.

Term, 11, 130.
Transposing, 86.

Unknown quantity, 84.

Variable, 63.

Zero, 223.
  multiplication by, 18.
  division by, 143.
  products, 202.

$R_1^2 + 2R_1R_2 + R_2^2 - (\quad)^2$

$2R_1R_2 = (\quad)$

$R_1 + R_2 = -\text{coef of } x$

$R_1R_2 = 3^{rd} \text{ Term}$

(c) (i) 17 (ii) $\sqrt{17}$ (iii) 33

$\frac{4}{3}$ $\frac{\sqrt{-176}}{3}$ $-\frac{80}{9}$

$\frac{3}{2}$ $\frac{7}{2}$ $\frac{29}{4}$